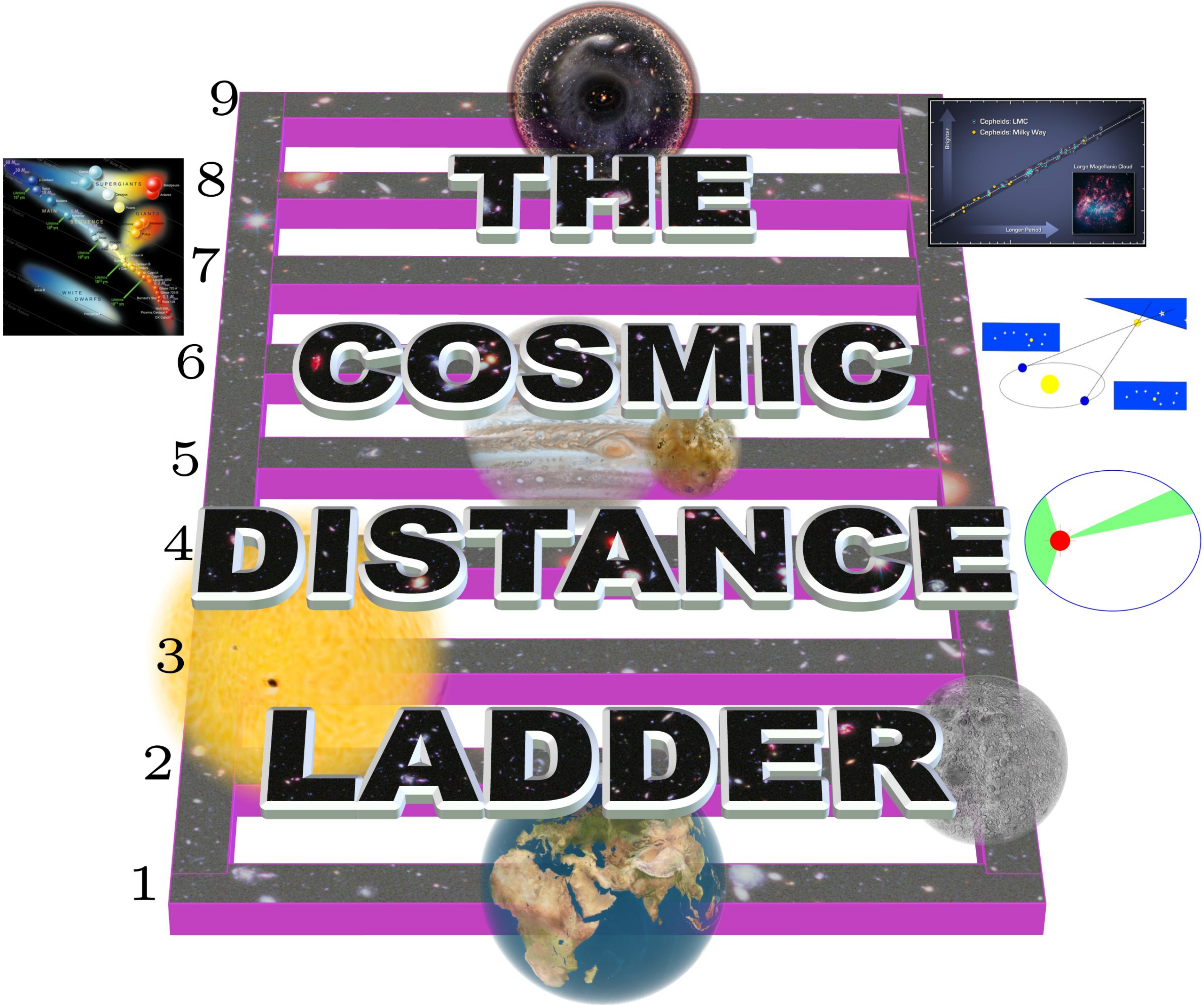
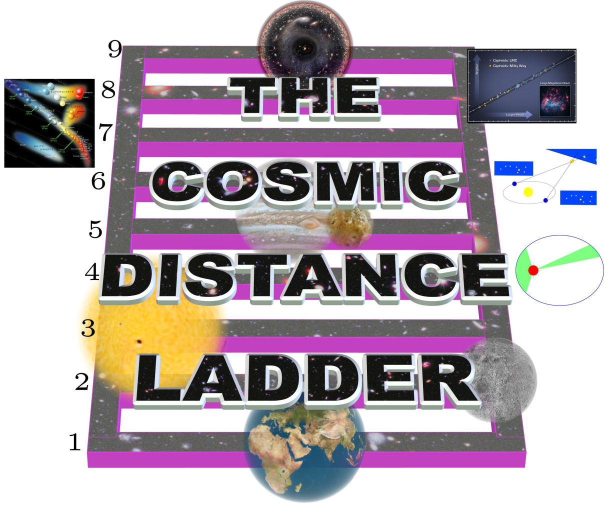
THE COSMIC DISTANCE LADDER



Dr Andrew French
2025

https://en.wikipedia.org/wiki/Cosmic_distance_ladder



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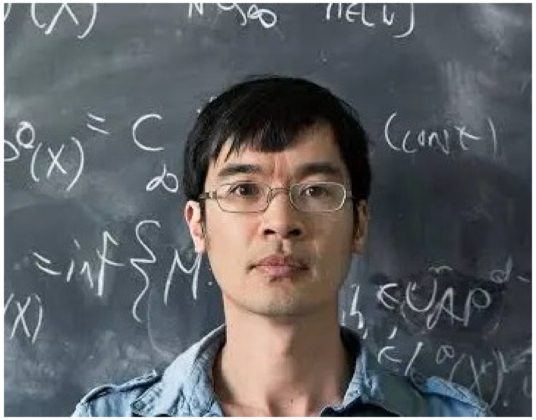
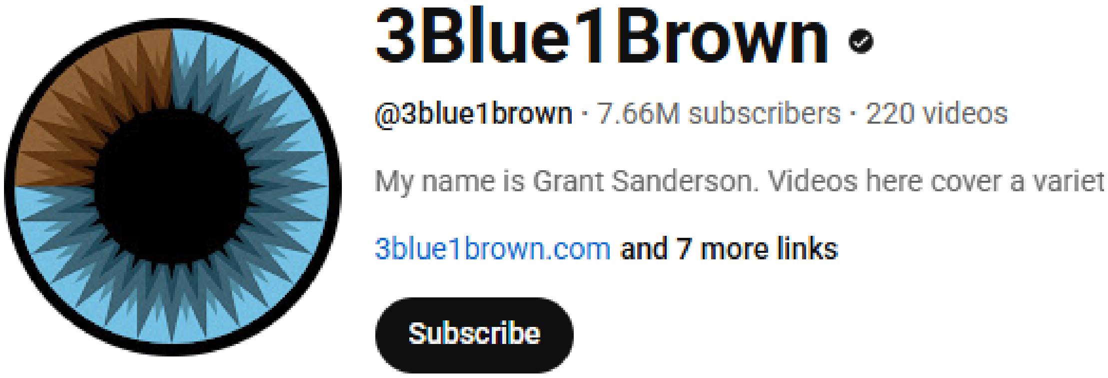


3Blue1 Brown

Terence Tao’s Cosmic Distance Ladder



3Blue1Brown

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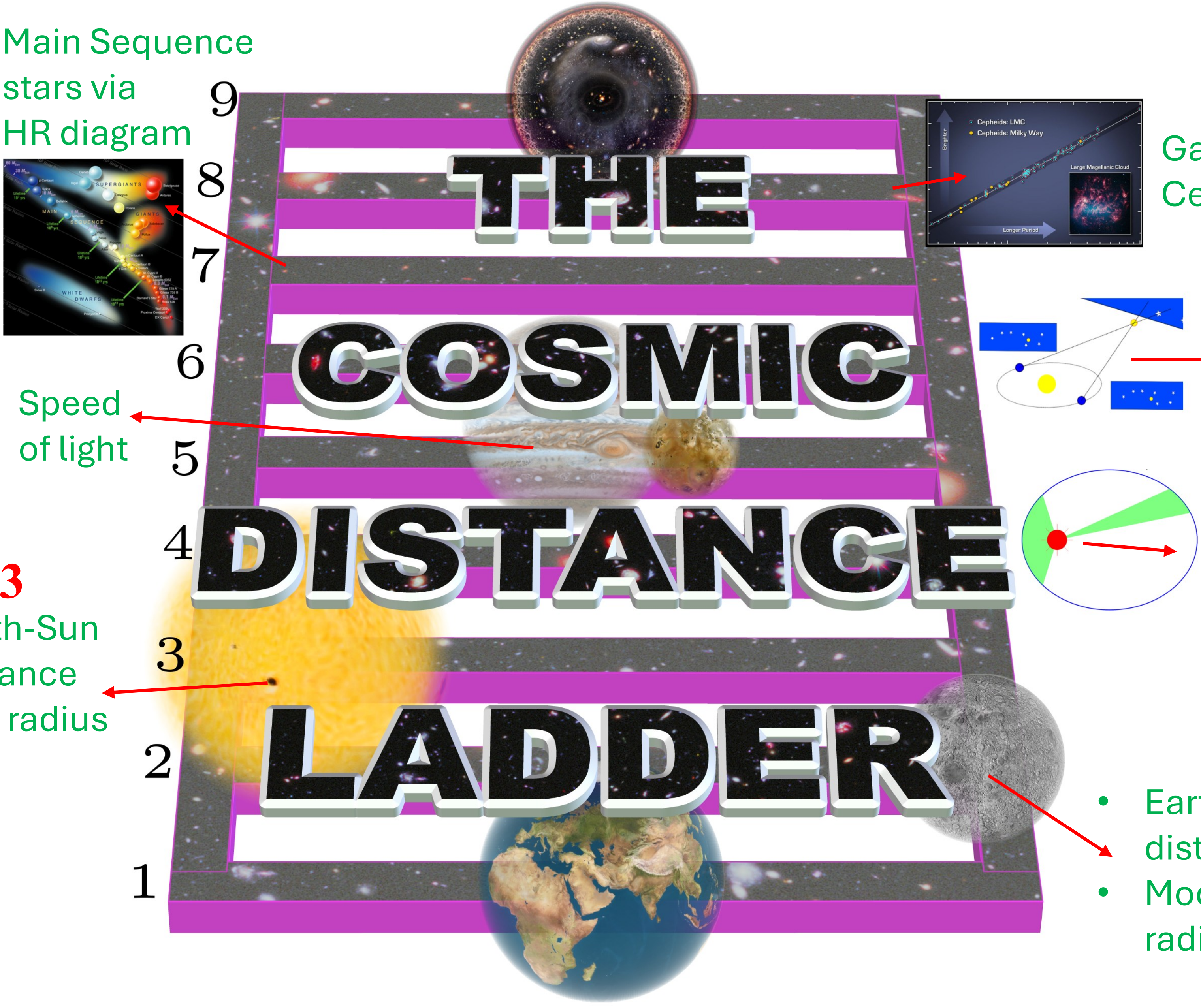
My name is Grant Sanderson. Videos here cover a variet

3blue1brown.com and 7 more links

9 Universe using Hubble law

7 Main Sequence stars via HR diagram



8 Galaxies via Cepheids

6 Nearby stars by parallax

5 Speed of light

4 Kepler's Laws, planetary orbits

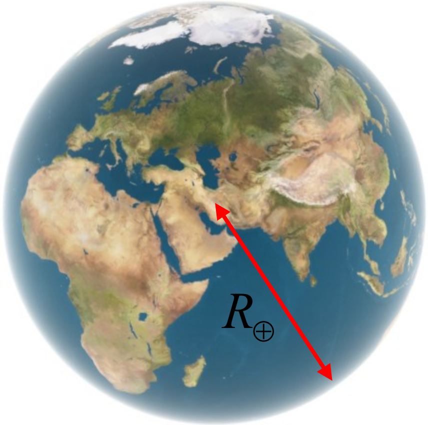
3

- Earth-Sun distance
- Sun radius

2

- Earth-Moon distance
- Moon radius

1 Calculating the radius of the Earth

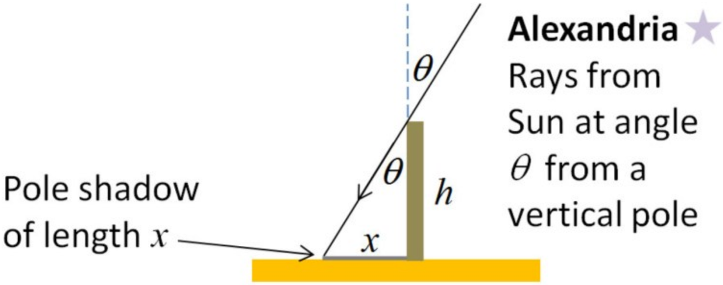


1. Calculating the radius of the Earth

$$\boxed{R_\oplus = 6378 \text{ km}}$$

Eratosthenes
(276BC-194BC)

$$\boxed{\theta \approx 7.8^{\text{o}}}$$



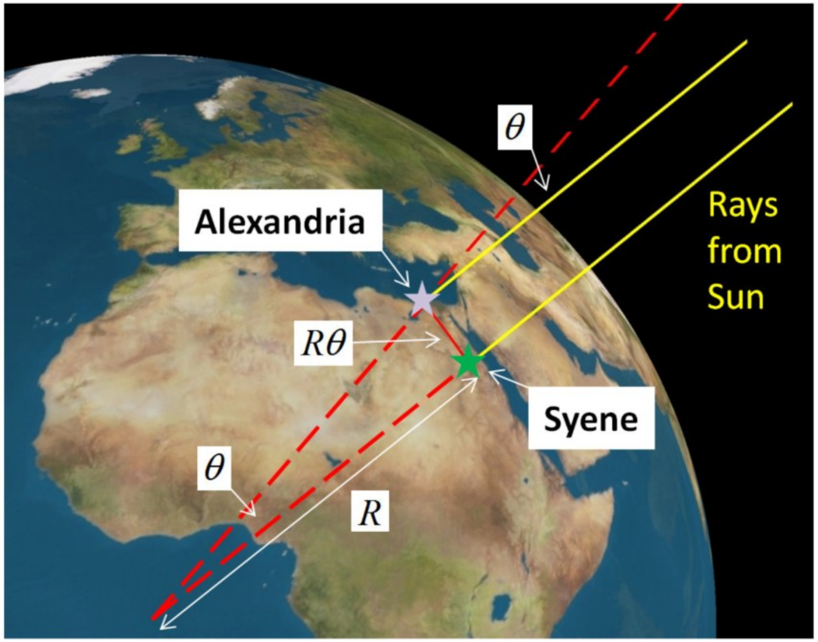
Alexandria ☆
Rays from
Sun at angle
θ from a
vertical pole

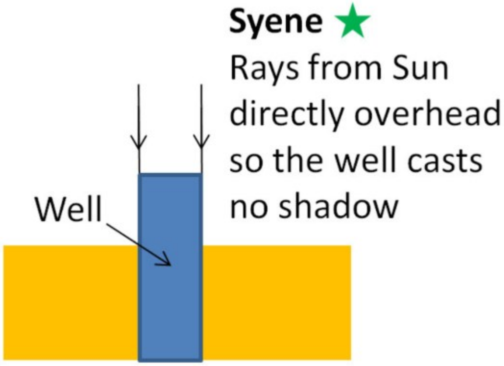
Pole shadow
of length x

$$R\theta \approx 5000 \times 185 \, \text{m}$$

$$\therefore R \approx \frac{5000 \times 185 \, \text{m}}{7.8 \times \frac{\pi}{180}} = \boxed{6.8 \times 10^6 \text{ m}}$$

6.6% error

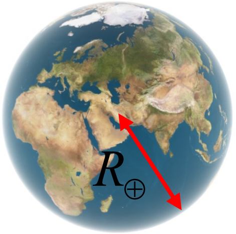




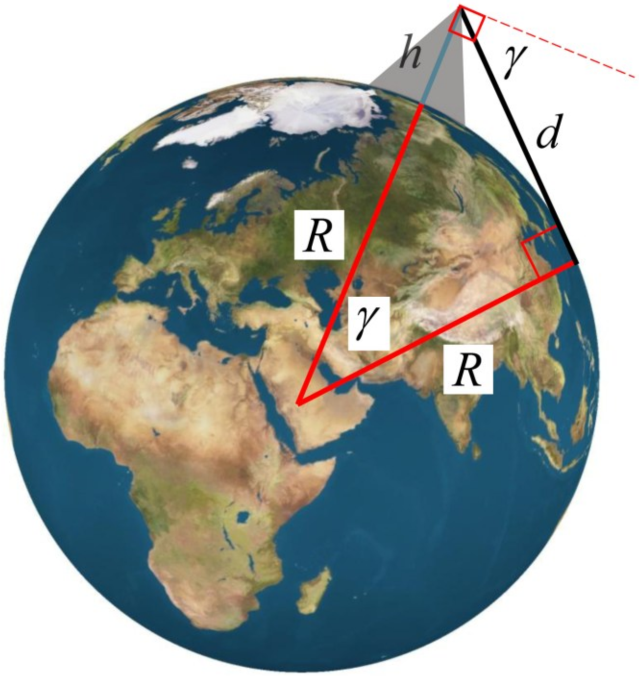
Syene ★
Rays from Sun
directly overhead
so the well casts
no shadow

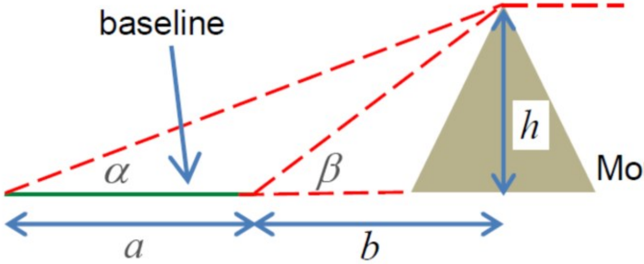
1. Calculating the radius of the Earth

$$\boxed{R_\oplus = 6378\ \text{km}}$$

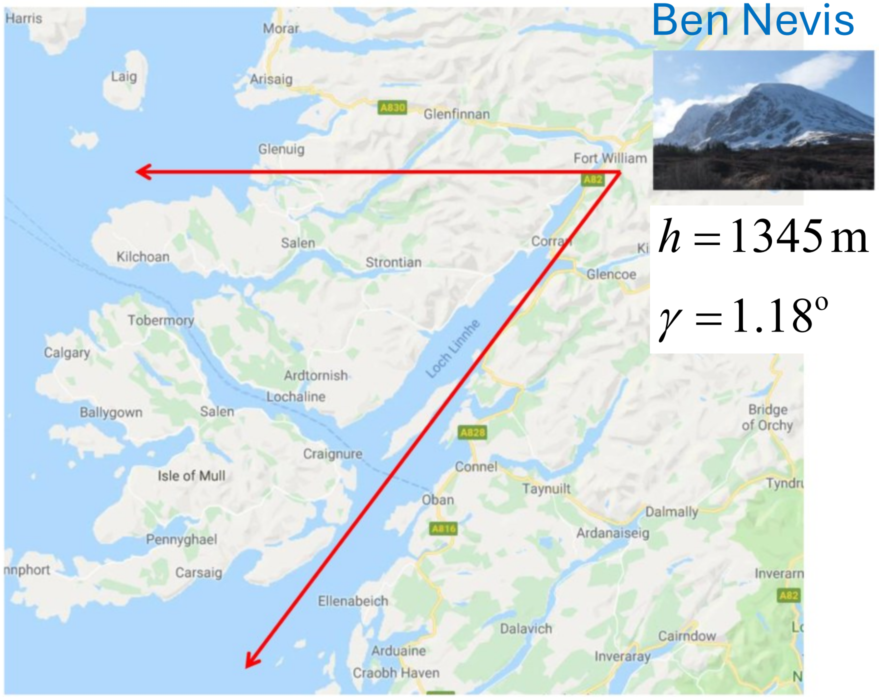


$$\boxed{h = \frac{\tan\alpha \tan\beta}{\tan\beta - \tan\alpha} a}$$





Ben Nevis



$$h = 1345\,\text{m}$$

$$\gamma = 1.18^{\circ}$$

Al-Biruni
(973-1050)

$$(R+h)\cos\gamma = R$$

$$\therefore \boxed{R = \frac{h\cos\gamma}{1-\cos\gamma}}$$

$$R_\oplus = \frac{1345\cos(1.18^{\circ})}{1-\cos(1.18^{\circ})} = \boxed{6340\,\text{km}}$$

0.6% error

Khwarazmian Iranian scholar
and polymath during the Islamic Golden Age

2. Calculating the distance of the moon from the earth, and the radius of the moon

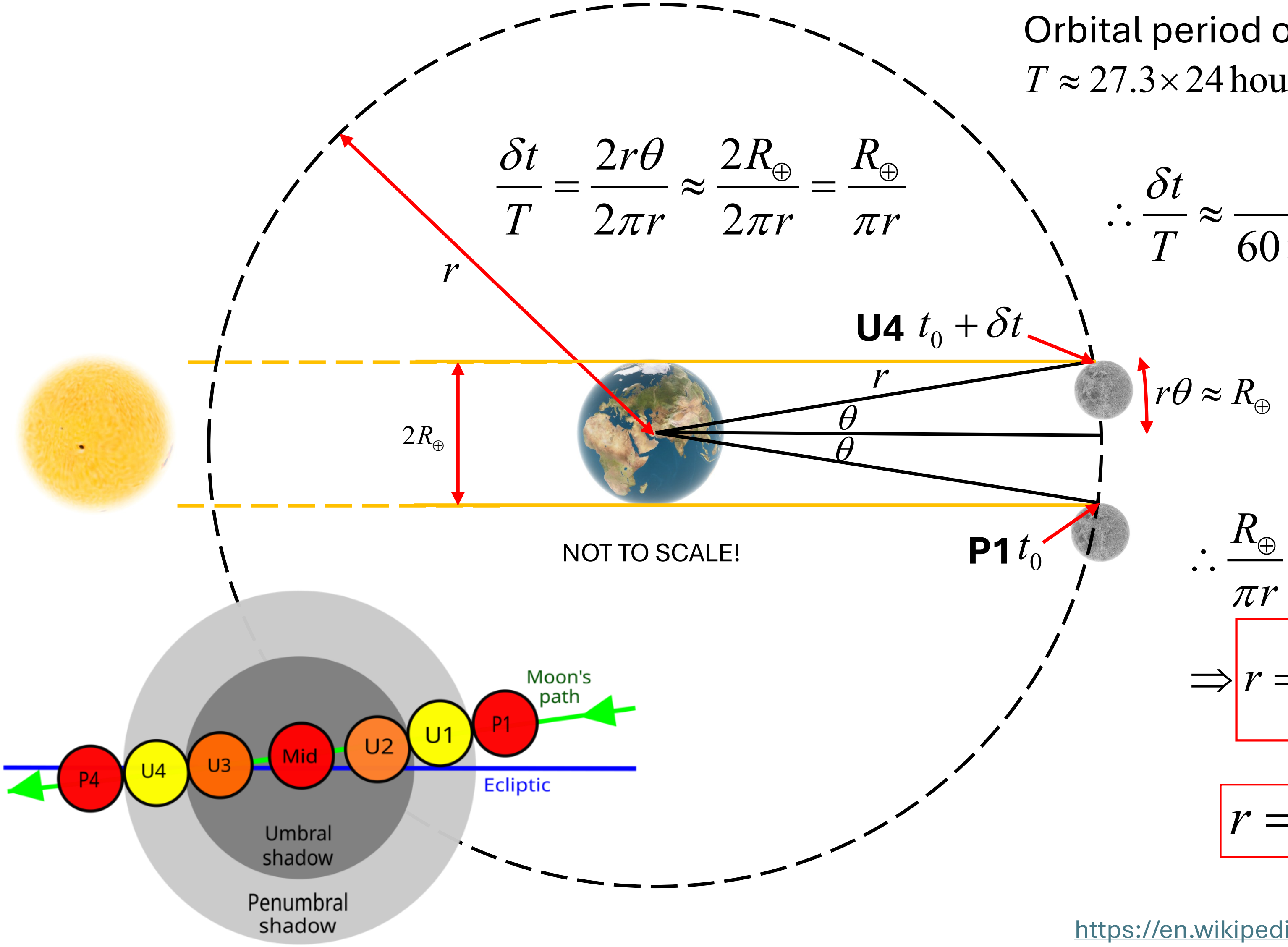
$$\boxed{R_\oplus = 6378 \text{ km}}$$

From **1.**

$\delta t \approx \frac{208}{60}$ hours is the *maximum* time of a lunar eclipse from **P1** to **U4**.

Orbital period of the Moon is
$T \approx 27.3 \times 24$ hours

$$\frac{\delta t}{T} = \frac{2r\theta}{2\pi r} \approx \frac{2R_\oplus}{2\pi r} = \frac{R_\oplus}{\pi r}$$



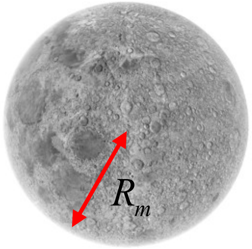
$$\therefore \frac{\delta t}{T} \approx \frac{208}{60 \times 27.3 \times 24} \approx \frac{1}{189}$$

$$\therefore \frac{R_\oplus}{\pi r} \approx \frac{1}{189}$$

$$\Rightarrow \boxed{r = \frac{189 R_\oplus}{\pi} \approx 60 R_\oplus}$$

$$\boxed{r = 384,400 \text{ km}}$$

https://en.wikipedia.org/wiki/Lunar_eclipse

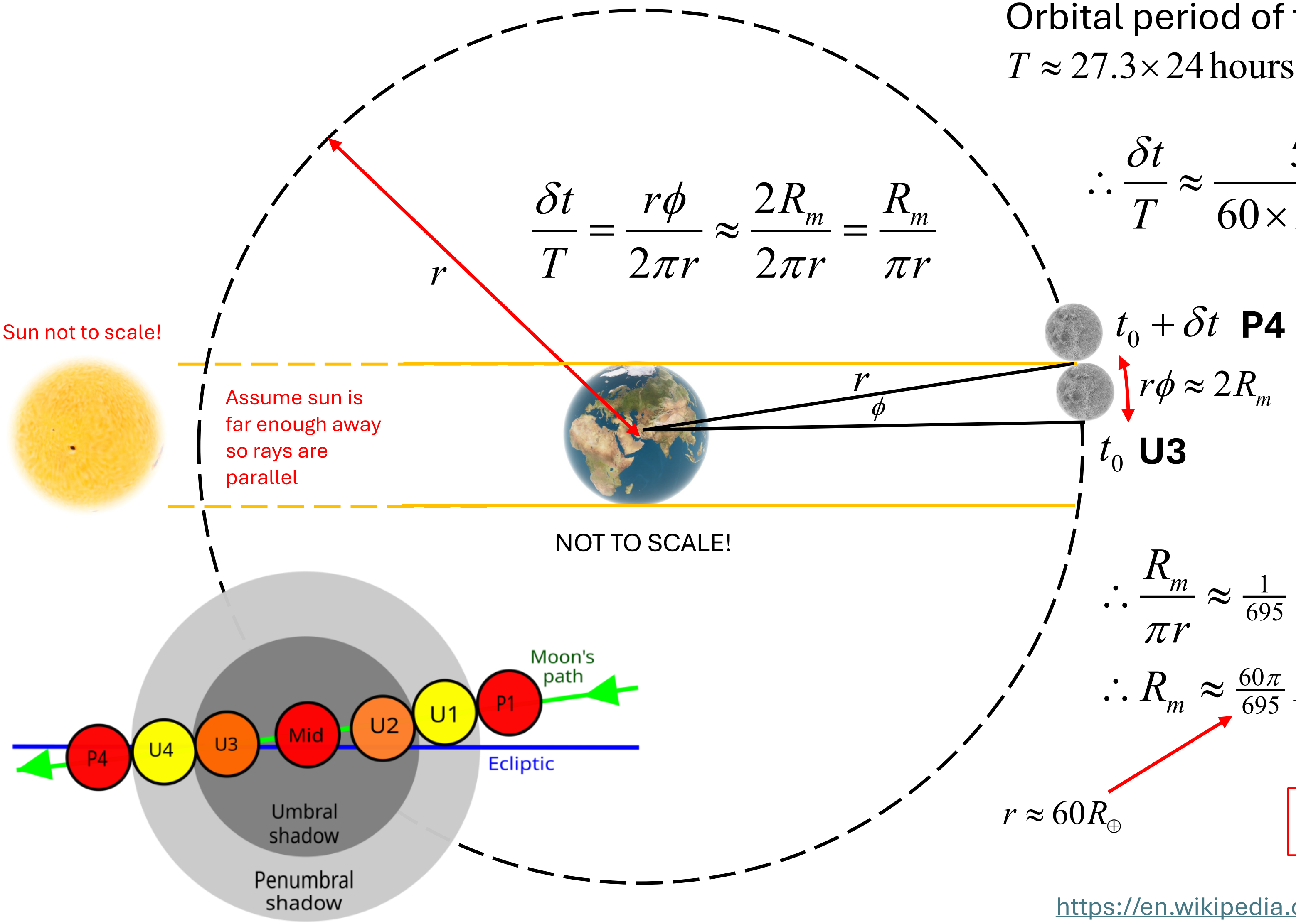


2. Calculating the distance of the moon from the earth, and the radius of the moon

$$\boxed{R_\oplus = 6378 \text{ km}}$$

From **1.**

$\delta t \approx \frac{56.6}{60}$ hours is the maximum time of a lunar eclipse from **U3** to **P4**.



$$\frac{\delta t}{T} = \frac{r\phi}{2\pi r} \approx \frac{2R_m}{2\pi r} = \frac{R_m}{\pi r}$$

Orbital period of the Moon is $T \approx 27.3 \times 24$ hours

$$\therefore \frac{\delta t}{T} \approx \frac{56.6}{60 \times 27.3 \times 24} \approx \tfrac{1}{695}$$

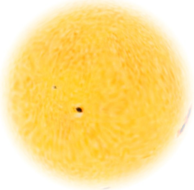
$$\therefore \frac{R_m}{\pi r} \approx \tfrac{1}{695} \Rightarrow R_m = \tfrac{1}{695}\pi r$$

$$\therefore R_m \approx \tfrac{60\pi}{695} R_\oplus \approx 0.27 R_\oplus$$

$r \approx 60R_\oplus$

$$\boxed{R_m = 1737 \text{ km}}$$

https://en.wikipedia.org/wiki/Lunar_eclipse



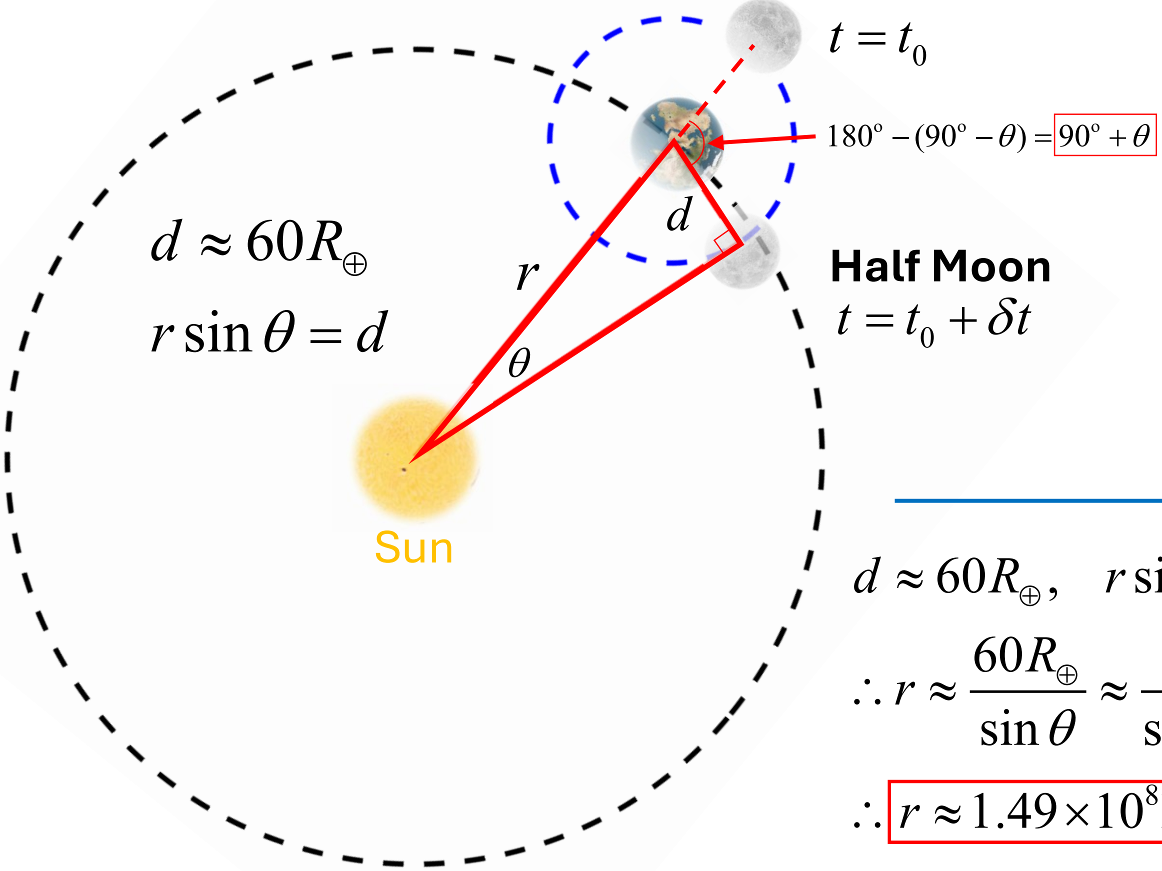
3. Calculating the Earth-Sun distance and the radius of the Sun

Orbital period of the Moon is $T \approx 27.3 \times 24\,\text{hours}$ $\quad \tfrac{1}{4}T = 163.8\,\text{hours}$

Precision required!

$$\boxed{\delta t = 164.068\,\text{hours}}$$

i.e. **16 min 3.1s** more than a quarter of an orbital period



$$\frac{\delta t}{T} = \frac{90^\circ + \theta}{360^\circ} \Rightarrow \theta = 360^\circ \frac{\delta t}{T} - 90^\circ$$

$$\therefore \theta = 360^\circ \frac{164.068}{27.3 \times 24} - 90^\circ = \boxed{0.147^\circ}$$

i.e. small

$$d \approx 60R_\oplus, \quad r \sin\theta = d$$

$$\therefore r \approx \frac{60R_\oplus}{\sin\theta} \approx \frac{60}{\sin(0.147^\circ)} = \boxed{23{,}386R_\oplus}$$

$$\therefore \boxed{r \approx 1.49 \times 10^8\,\text{km}}$$

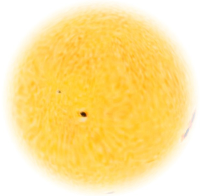
$\boxed{R_\oplus = 6378\ \text{km}}$ From **1.**

$\boxed{d \approx 60R_\oplus = 384{,}400\,\text{km}}$ From **2.**

2012 definition of an **Astronomical Unit**

$$\boxed{1\text{AU} = 149{,}597{,}870{,}700\ \text{m} \approx \overline{r_{\odot\oplus}}}$$

Aristarchus of Samos (310-230BC)



3. Calculating the Earth-Sun distance and the radius of the Sun

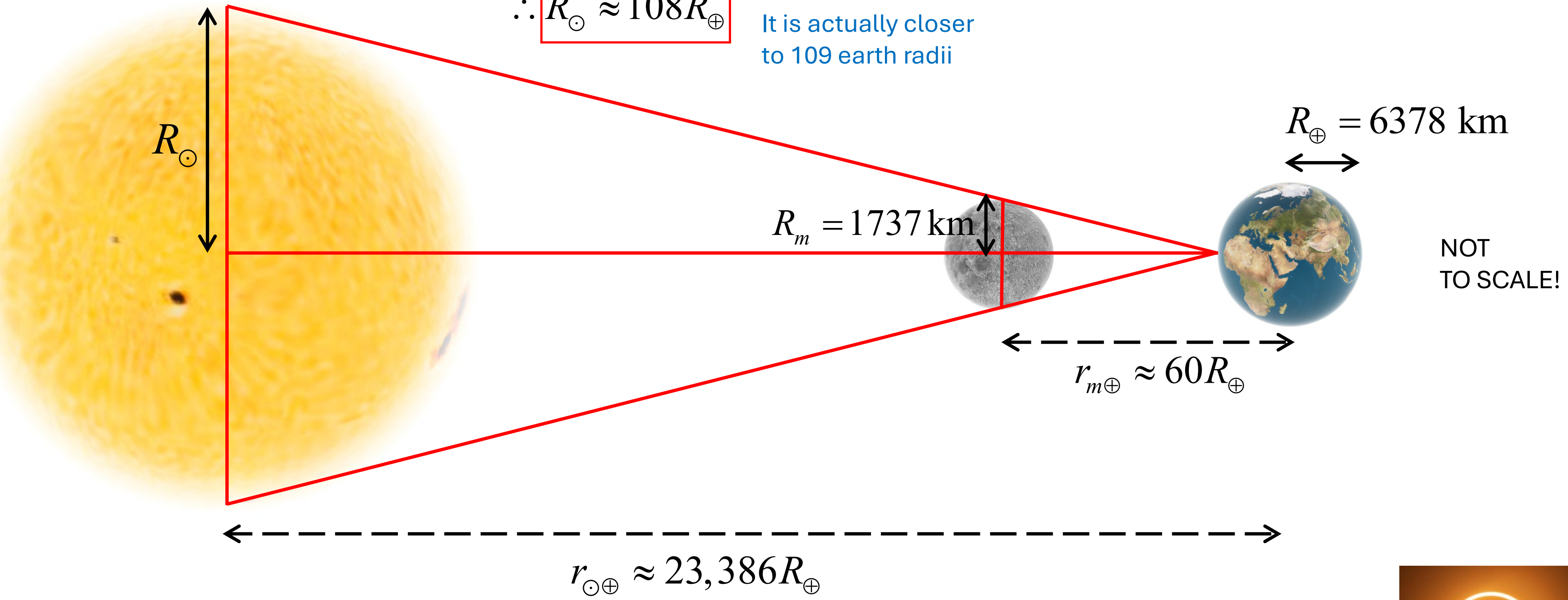
During a **total solar eclipse**, the moon obscures the Sun almost exactly*. Hence by similar triangles:

$$\frac{R_\odot}{23{,}385R_\oplus} = \frac{R_m}{59R_\oplus}$$

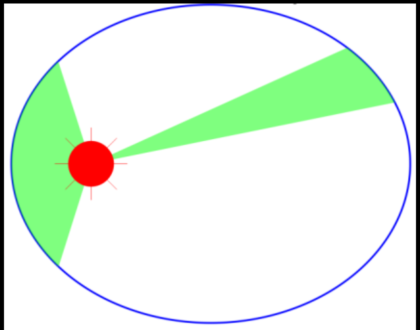
$$\therefore R_\odot \approx \tfrac{23{,}385}{59}\times 1737\,\text{km} \approx 688{,}470\,\text{km}$$

$$\therefore \boxed{R_\odot \approx 108R_\oplus}$$

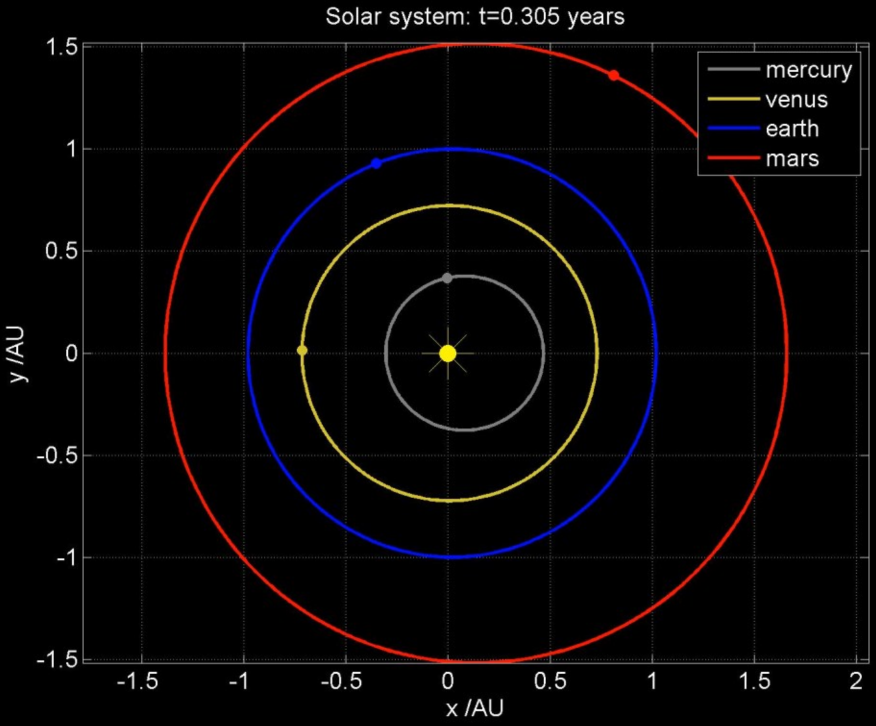
It is actually closer to 109 earth radii

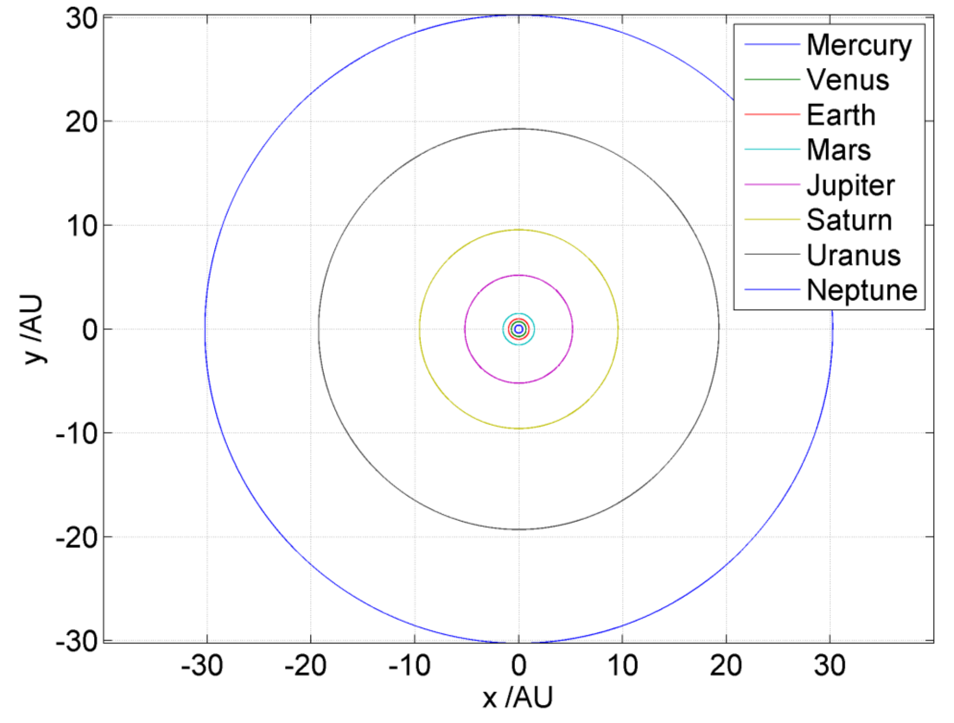


*The Moon's orbit is slightly elliptical, meaning the Earth-Moon distance varies from 57 to 64 Earth radii. This means you can get an annular eclipse, which means a 'fiery ring' around the moon.

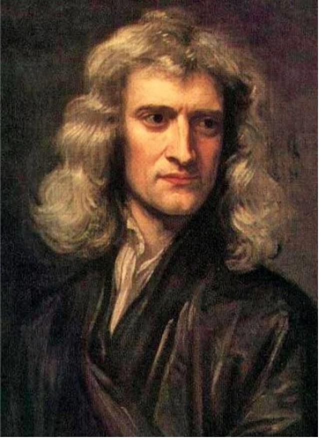


4. Calculating the orbits of the planets in the Solar System

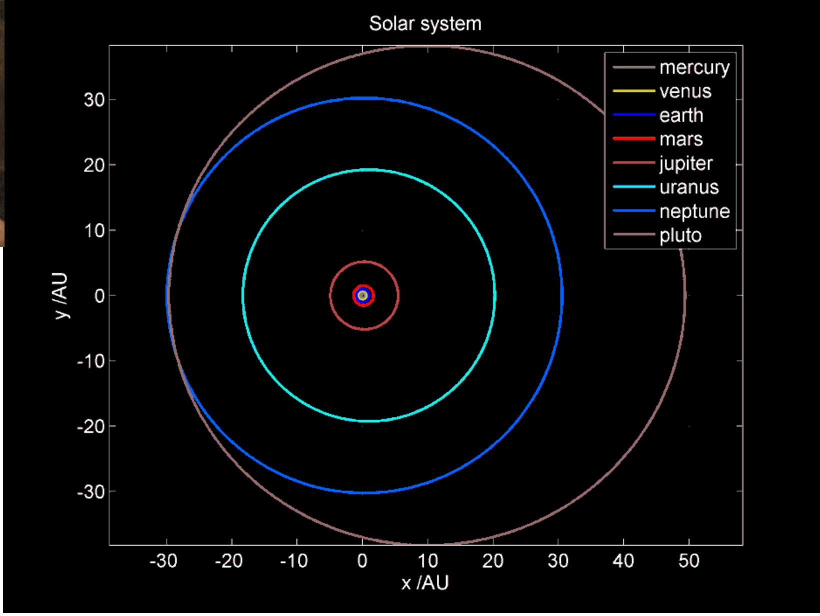




The Solar System to Scale



Isaac Newton (1642-1727) developed a mathematical model of Gravity which predicted the ***elliptical*** orbits proposed by Johannes Kepler (1571-1630)



Planet and star masses

Force of gravity

$$F = \frac{GmM}{r^2}$$

Universal gravitational constant

$$G = 6.67 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$$

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

Polar equation of ellipse

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

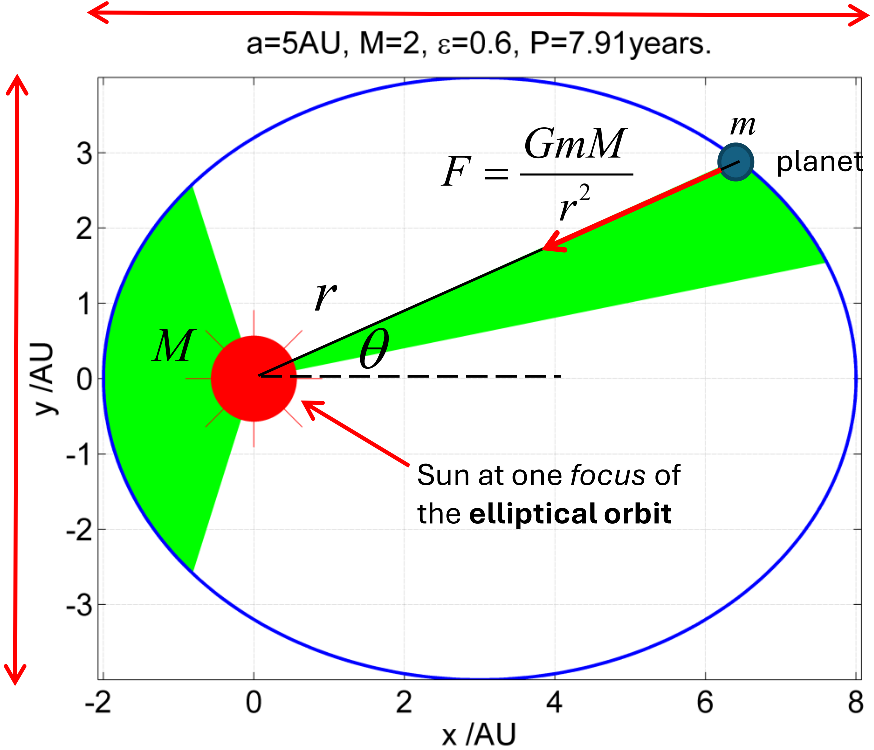
Eccentricity of ellipse

$$P^2 = \frac{4\pi^2}{G(m+M)}a^3$$

Orbital period P

$2a$ ← Semi-major axis

Semi-minor axis ↘ $2b$



Kepler's three laws are:

$G = 6.67\times10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$ $\quad M_\odot = 1.9891\times10^{30}\,\mathrm{kg}$

1. *The orbit of every planet in the solar system is an ellipse with the Sun at one of the two foci.*
2. *A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.*
3. *The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.*

The wording of Kepler's laws implies a specific application to the solar system. However, the laws are more generally applicable to any system of two masses whose mutual attraction is an inverse-square law.

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

Polar equation of ellipse

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

Eccentricity of ellipse

$$P^2 = \frac{4\pi^2}{G(m+M)}a^3$$

Orbital period P

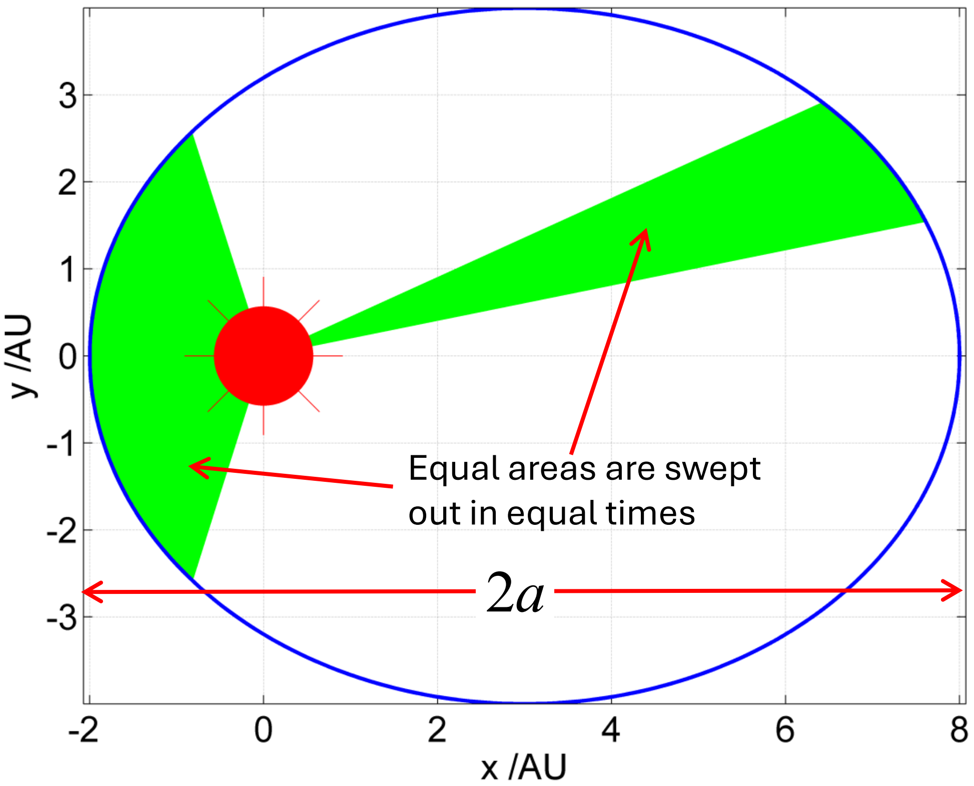
Planet mass

Sun mass

$$\frac{dA}{dt} = \tfrac{1}{2}\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}$$

This is a *constant* of the orbit

a=5AU, M=2, ε=0.6, P=7.91years.



Equal areas are swept out in equal times

Johannes Kepler
1571-1630

Object	Mass in Earth masses	Distance from Sun in AU	Radius in Earth radii	Rotational period /days	Orbital period /years	Gravitational field (in terms of g = 9.81 ms^-2)
Saturn	95.16	9.58	9.45	0.44	29.63	1.07
Uranus	14.50	19.29	4.01	0.72	84.75	0.90
Jupiter	317.85	5.20	11.21	0.41	11.86	2.53
Sun	332,837	-	109.12	-	-	27.95
Neptune	17.20	30.25	3.88	0.67	166.34	1.14
Pluto	0.00	39.51	0.19	6.39	248.35	0.09
Mars	0.107	1.523	0.53	1.03	1.88	0.38
Venus	0.815	0.723	0.95	243.02	0.62	0.90
Mercury	0.055	0.387	0.38	58.65	0.24	0.37
Earth	1.000	1.000	1.00	1.00	1.00	1.00

For *our* Solar System:

$$P^2 = \frac{4\pi^2}{G\left(m + M_\odot\right)} a^3$$

$m \ll M_\odot$

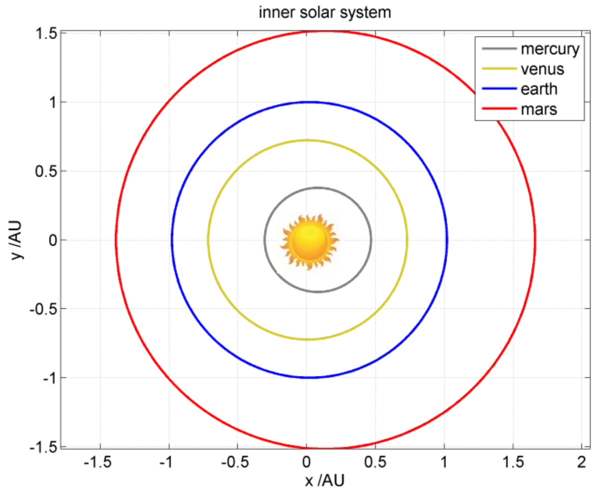
$$P^2 \approx \frac{4\pi^2}{GM_\odot} a^3$$

$$G = 6.67 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$$

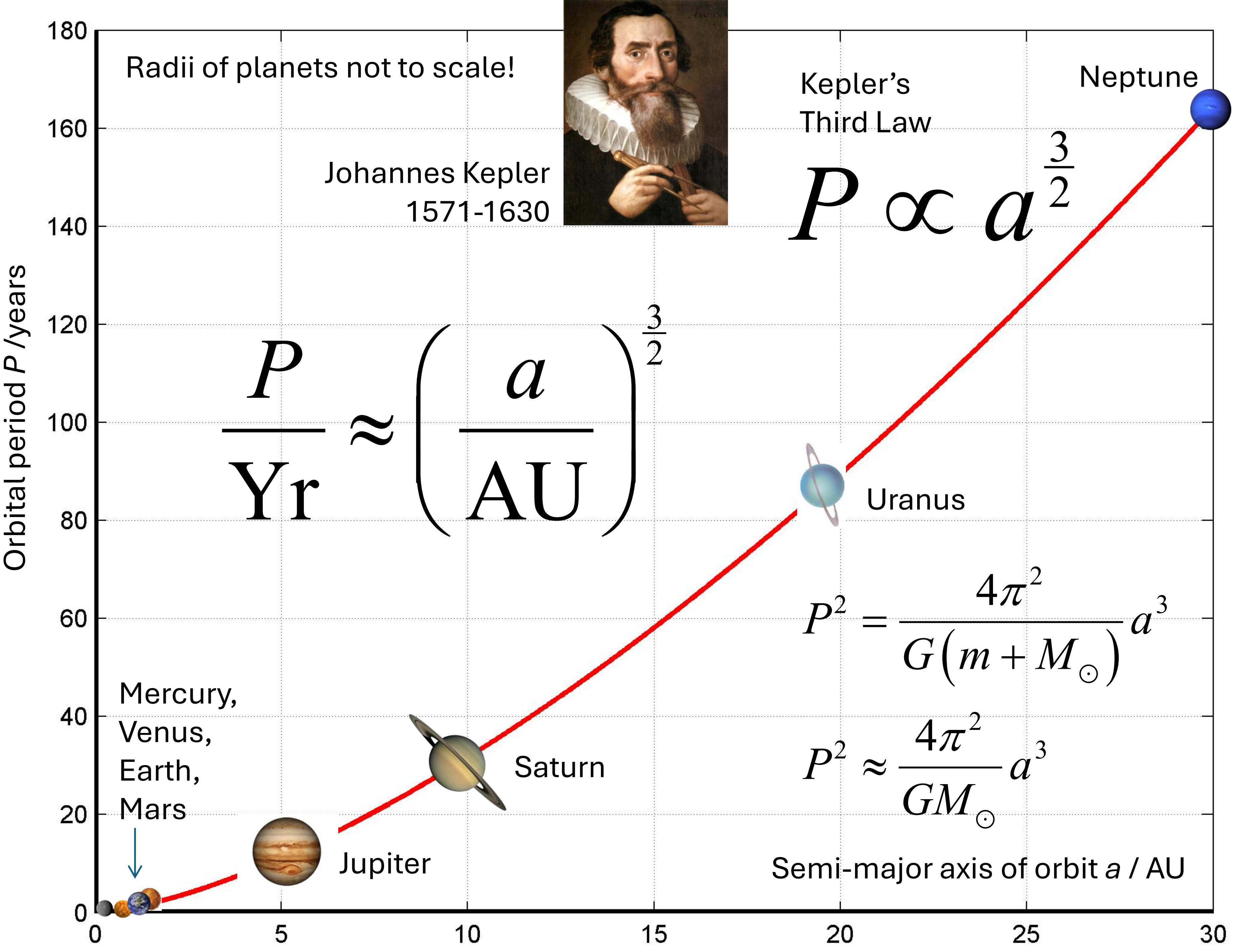
$$M_\odot = 1.9891 \times 10^{30}\,\mathrm{kg}$$

$$\mathrm{Yr}^2 = \frac{4\pi^2}{GM_\odot} \mathrm{AU}^3$$

$$\boxed{\therefore \frac{P}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}}\right)^{\frac{3}{2}}}$$



$$1\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$$



Kepler's Third Law

$$P \propto a^{\frac{3}{2}}$$

$$\frac{P}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}} \right)^{\frac{3}{2}}$$

$$P^2 = \frac{4\pi^2}{G\left(m + M_\odot\right)} a^3$$

$$P^2 \approx \frac{4\pi^2}{GM_\odot} a^3$$

An improved calculation of the Earth-Sun distance using the transit of Venus

$$D \approx \left(r_{\odot\oplus} - r_{\odot V}\right)\theta, \quad r_{\odot\oplus} \approx 1\text{AU}, \; r_{\odot V} \approx 0.723\text{AU}$$

From **Kepler III** and orbital period of Venus of 225 days.

$$\therefore 1\text{AU} = \frac{D}{0.277\theta}$$

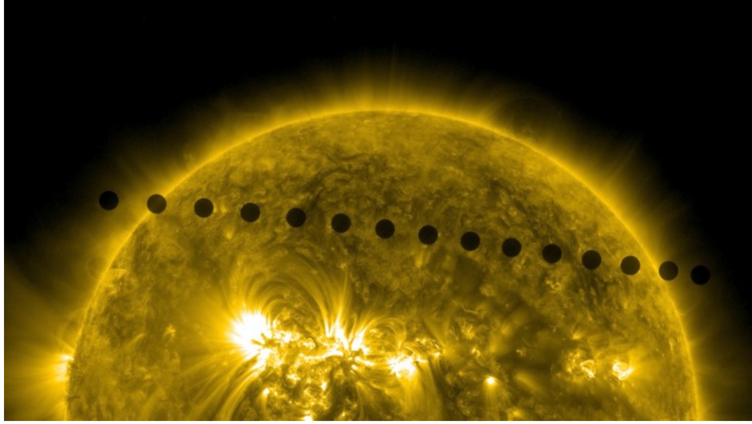
$$D \approx R_{\oplus} = 6378 \text{ km}$$

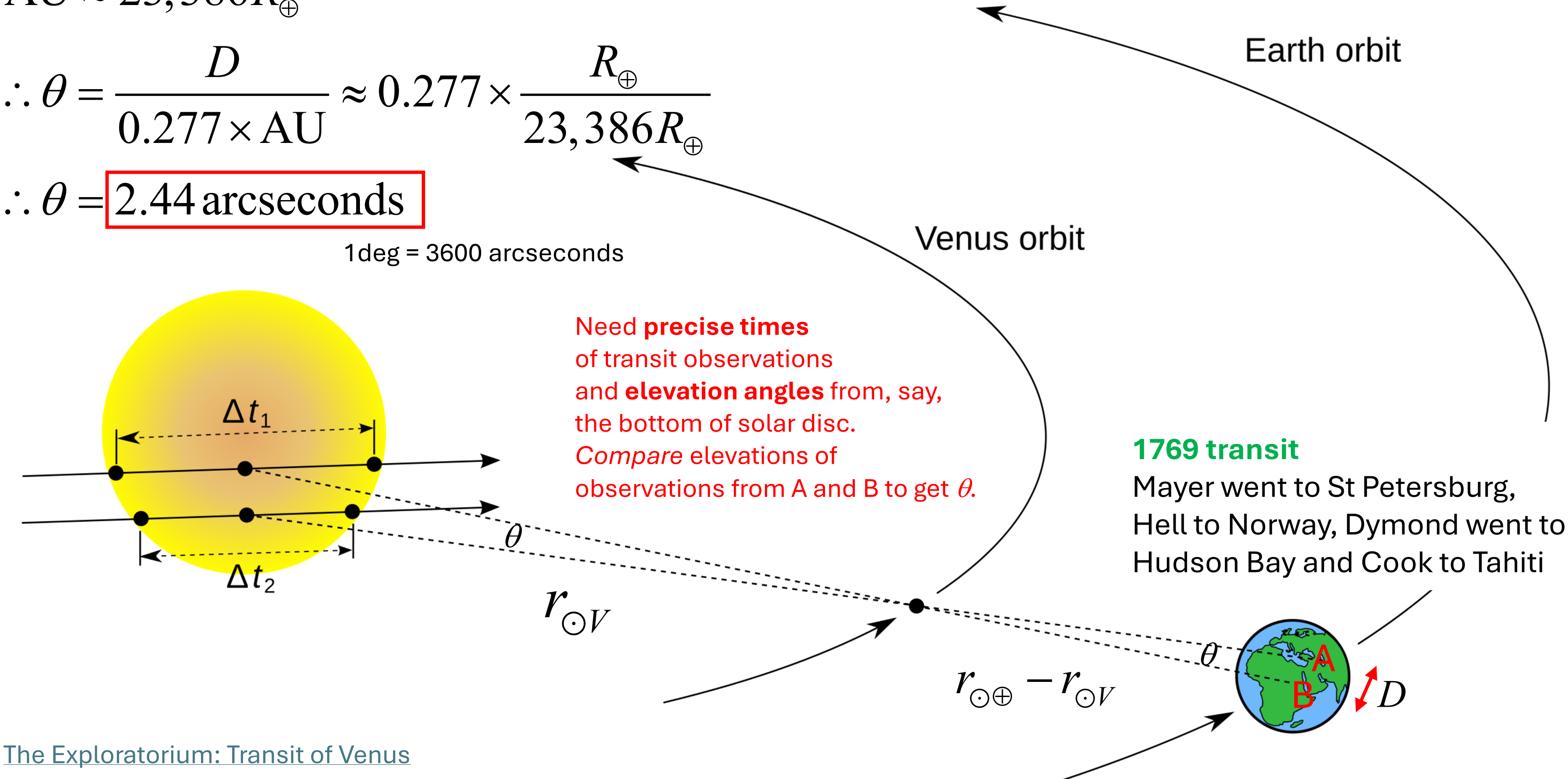
$$\text{AU} \approx 23{,}386 R_{\oplus}$$

$$\therefore \theta = \frac{D}{0.277 \times \text{AU}} \approx 0.277 \times \frac{R_{\oplus}}{23{,}386 R_{\oplus}}$$

$$\therefore \theta = \boxed{2.44 \,\text{arcseconds}}$$

1deg = 3600 arcseconds





Need **precise times** of transit observations and **elevation angles** from, say, the bottom of solar disc. *Compare* elevations of observations from A and B to get θ.

1769 transit

Mayer went to St Petersburg, Hell to Norway, Dymond went to Hudson Bay and Cook to Tahiti

The Exploratorium: Transit of Venus

Object	$M/M_\oplus$	a /AU	ε	θ_0	β
Sun	332,837	-	-	-	-
Mercury	0.055	0.387	0.21	*	7.00
Venus†	0.815	0.723	0.01	*	3.39
Earth	1.000	1.000	0.02	*	0.00
Mars	0.107	1.523	0.09	*	1.85
Jupiter	317.85	5.202	0.05	*	1.31
Saturn	95.159	9.576	0.06	*	2.49
Uranus†	14.500	19.293	0.05	*	0.77
Neptune	17.204	30.246	0.01	*	1.77
Pluto†	0.003	39.509	0.25	*	17.5

$R/R_\oplus$	T_{rot} / days	P /Yr
109.123	-	-
0.383	58.646	0.241
0.949	243.018	0.615
1.000	0.997	1.000
0.533	1.026	1.881
11.209	0.413	11.861
9.449	0.444	29.628
4.007	0.718	84.747
3.883	0.671	166.344
0.187	6.387	248.348

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

$$P^2 = \frac{4\pi^2}{G(m+M_\odot)}a^3$$

β is the orbital inclination /degrees. In all cases the semi-major axis pointing direction is

$$\mathbf{d} = d_x\hat{\mathbf{x}} + d_y\hat{\mathbf{y}} + d_{\mathbf{y}}\hat{\mathbf{z}} = \cos\beta\hat{\mathbf{x}} + \sin\beta\hat{\mathbf{z}}$$

You could begin with all zero, or perhaps a random angle for each planet's orbit.

$$\begin{aligned}
M_\odot &= 1.9891 \times 10^{30} \text{ kg} \\
R_\odot &= 6.960 \times 10^{8} \text{ m} \\
M_\oplus &= 5.9742 \times 10^{24} \text{ kg} \\
R_\oplus &= 6.37814 \times 10^{6} \text{ m} \\
1\text{AU} &= 1.495979 \times 10^{11} \text{ m}
\end{aligned}$$

* For the current orbital polar angle θ_0 (and indeed more accurate values for solar system parameters) see the website of the Jet Propulsion Laboratory (JPL) `http://ssd.jpl.nasa.gov/`

†These planets rotate clockwise about their own internal polar axis. ("Retrograde"). All the other planets rotate anti-clockwise about their own internal axis. All the planets orbit the sun in an anticlockwise direction.

Calculating orbit angle vs time

$$\boxed{\frac{dA}{dt}=\tfrac{1}{2}\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}}$$

This is a *constant*

Orbit time can be determined from polar angle using Kepler II:

$$r^2\frac{d\theta}{dt}=\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}$$

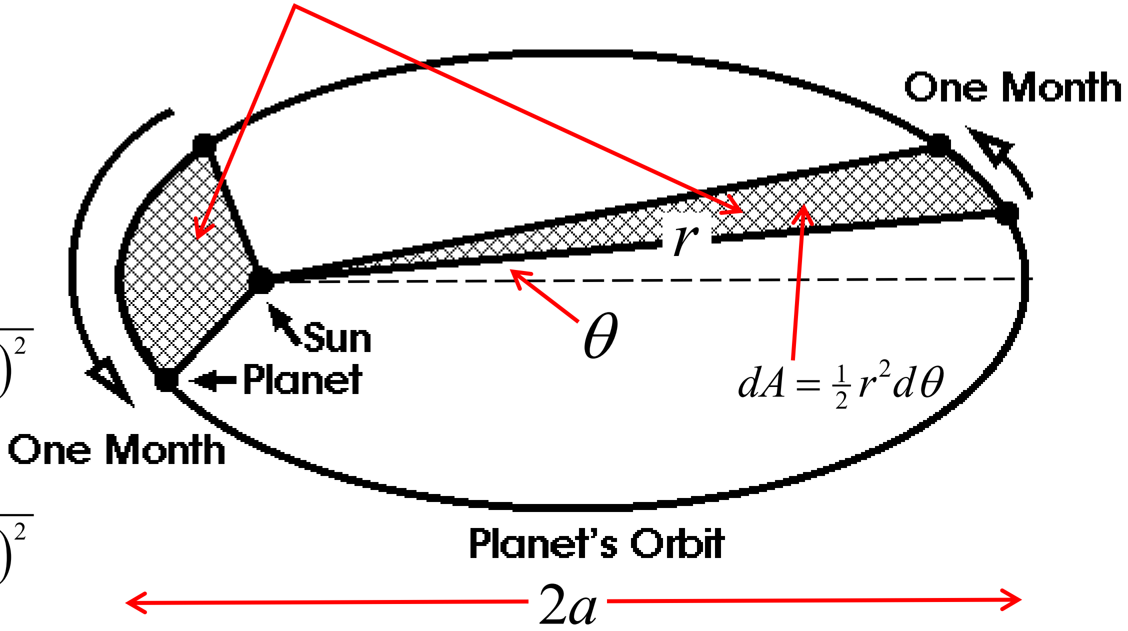
$$\therefore\int_{\theta_0}^{\theta} r^2 d\theta = t\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}$$

$$\therefore t=\frac{a^2\left(1-\varepsilon^2\right)^2}{\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}}\int_{\theta_0}^{\theta}\frac{d\theta}{\left(1-\varepsilon\cos\theta\right)^2}$$

$$\therefore t=\frac{a^2\left(1-\varepsilon^2\right)^2}{\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}}\int_{\theta_0}^{\theta}\frac{d\theta}{\left(1-\varepsilon\cos\theta\right)^2}$$

$$\therefore t=\sqrt{\frac{a^3\left(1-\varepsilon^2\right)^3}{G(m+M)}}\int_{\theta_0}^{\theta}\frac{d\theta}{\left(1-\varepsilon\cos\theta\right)^2}$$

Equal areas swept out in equal times



From Kepler III: $P^2=\dfrac{4\pi^2}{G(m+M)}a^3$

$$\boxed{t=P\left(1-\varepsilon^2\right)^{\frac{3}{2}}\tfrac{1}{2\pi}\int_{\theta_0}^{\theta}\frac{d\theta}{\left(1-\varepsilon\cos\theta\right)^2}}$$

Evaluate this *numerically*

Note when: $\boxed{\begin{array}{l}\varepsilon\ll 1\\ t\approx P\left(\theta-\theta_0\right)\end{array}}$

How Kepler determined the elliptical orbit of Mars (and indeed the other planets)

Johannes Kepler
1571-1630

$$\tan\phi = \frac{y-\sin\theta}{x-\cos\theta}$$
$$\therefore y = x\tan\phi - \tan\phi\cos\theta + \sin\theta$$

$$T \approx 1.523^{1.5}\ \text{yr} = 1.88\ \text{yr}\ \ (687\ \text{days})$$

Mars orbital period

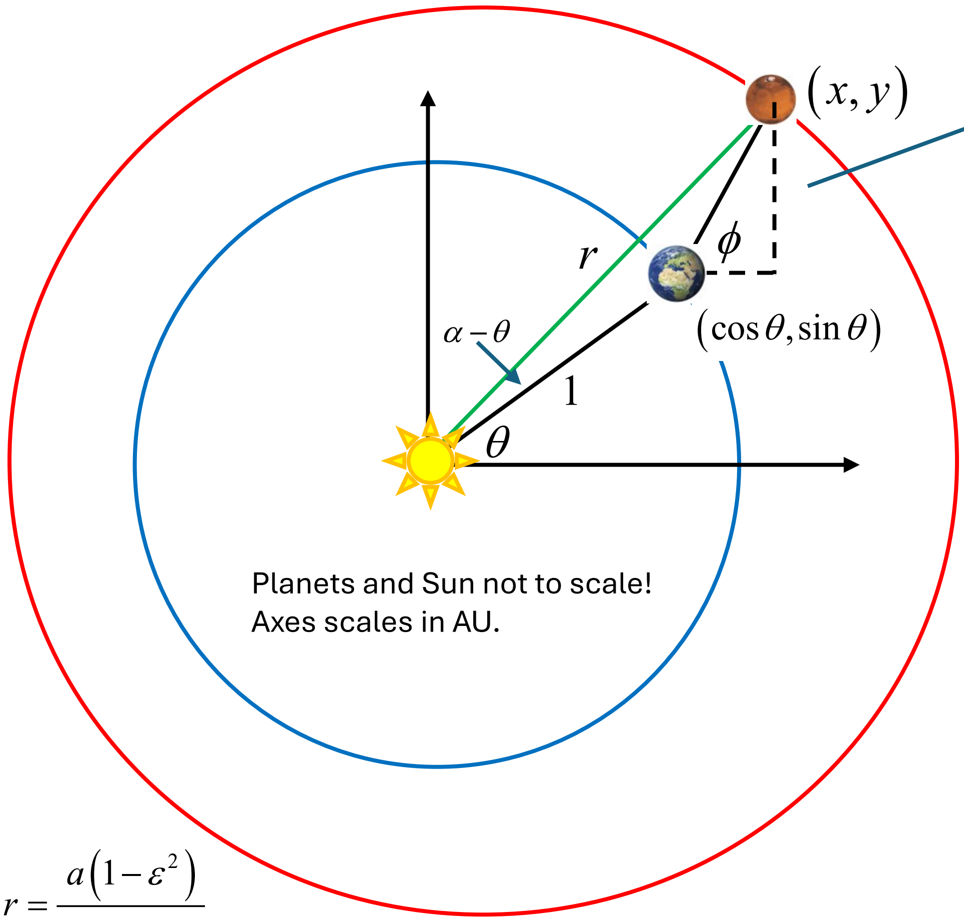
ATTEMPT #1: Assume Earth's orbit about the Sun is *circular*, with radius approximately 1AU.

Measure angles θ and ϕ at times separated by one Mars orbital period T. Hence (x,y) coordinates of Mars *should remain the same.*

$$y = x\tan\phi_1 - \tan\phi_1\cos\theta_1 + \sin\theta_1$$
$$y = x\tan\phi_2 - \tan\phi_2\cos\theta_2 + \sin\theta_2$$
$$\theta_1 = \theta(t),\ \theta_2 = \theta(t+T)$$
$$\phi_1 = \phi(t),\ \phi_2 = \phi(t+T)$$

$$\therefore x = \frac{\sin\theta_1 - \sin\theta_2 - \tan\phi_1\cos\theta_1 + \tan\phi_2\cos\theta_2}{\tan\phi_2 - \tan\phi_1}$$
$$\therefore y = x\tan\phi_1 - \tan\phi_1\cos\theta_1 + \sin\theta_1$$

Use this expression to work out orbit of Mars using the angles θ and ϕ. Note the latter could have been measured from Earth since ancient times!



Planets and Sun not to scale!
Axes scales in AU.

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\alpha}$$
$$x = r\cos\alpha$$
$$y = r\sin\alpha$$

Exact orbit using Kepler's laws
Mars: $a = 1.523,\ \varepsilon = 0.09.$

$$t = T\left(1-\varepsilon^2\right)^{\frac{3}{2}}\frac{1}{2\pi}\int_{\alpha_0}^{\alpha}\frac{d\lambda}{\left(1-\varepsilon\cos\lambda\right)^2}$$

Use this to generate simulation data!

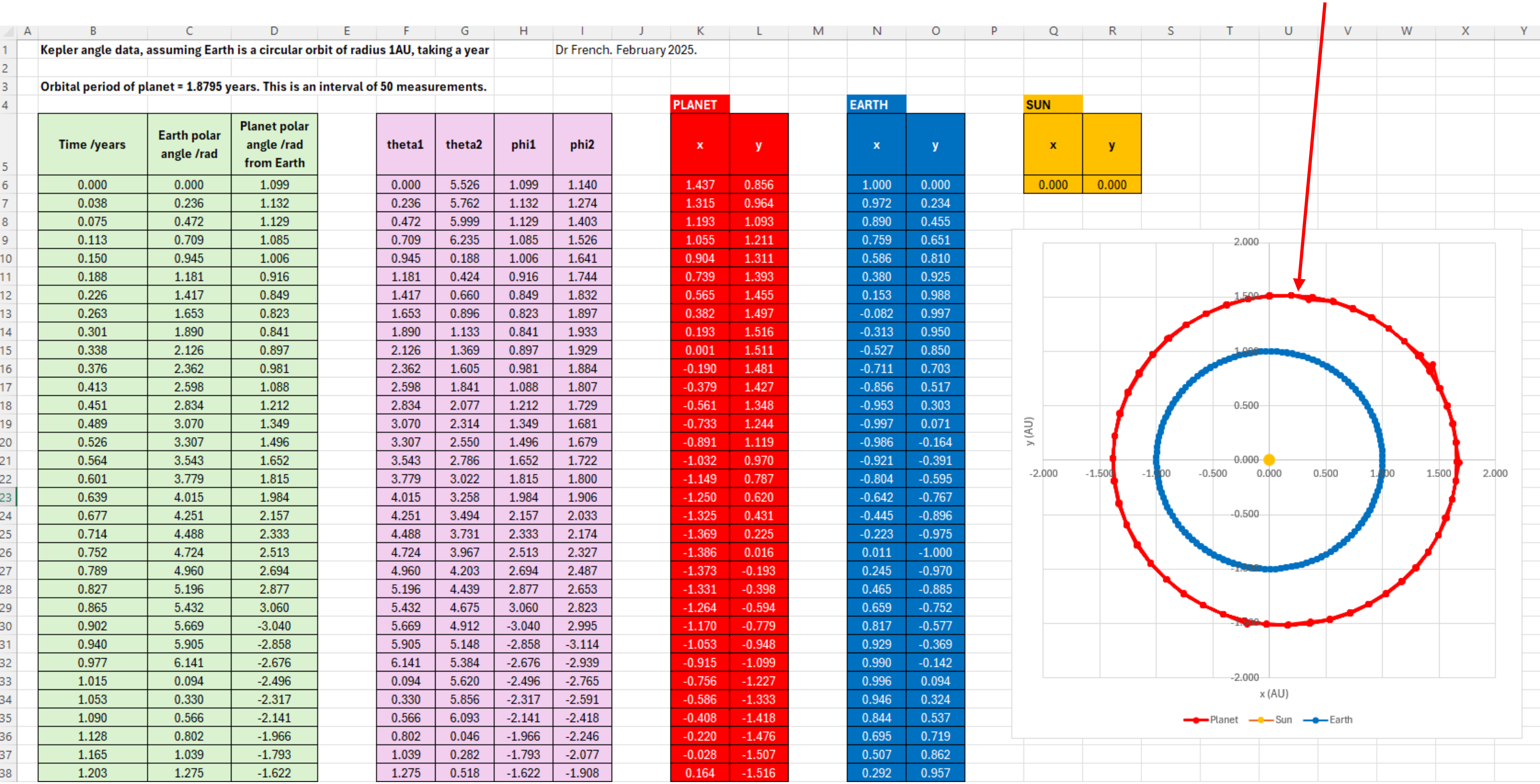
Terence Tao: The Cosmic Distance Ladder Part 1 (3Blue1Brown)

Astronomical Unit (AU) *defined* in 2012 $1\text{AU} = 149{,}597{,}870{,}700\ \text{m}$

$$\therefore x = \frac{\sin\theta_1 - \sin\theta_2 - \tan\phi_1\cos\theta_1 + \tan\phi_2\cos\theta_2}{\tan\phi_2 - \tan\phi_1}$$

$$\therefore y = x\tan\phi_1 - \tan\phi_1\cos\theta_1 + \sin\theta_1$$

Best fit to Mars' orbit is an *ellipse*.
This justifies **Kepler's First law**



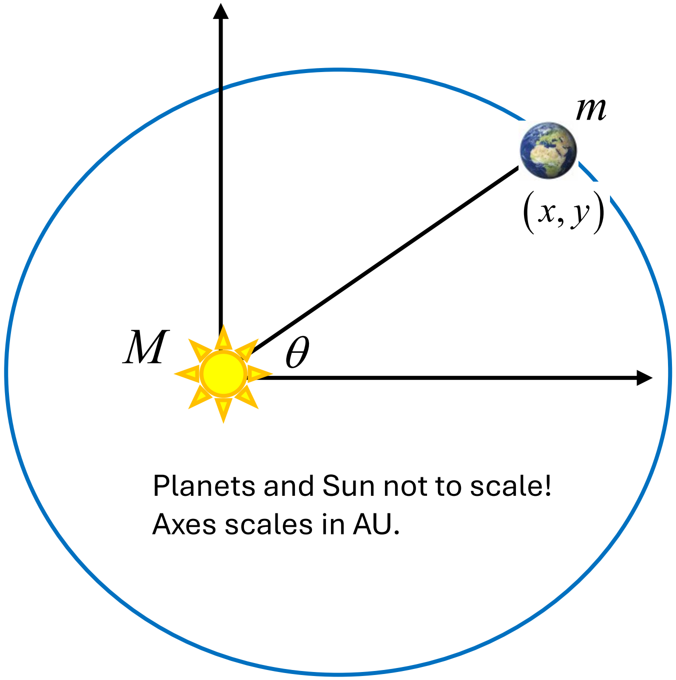
Kepler angle data, assuming Earth is a circular orbit of radius 1AU, taking a year Dr French. February 2025.

Orbital period of planet = 1.8795 years. This is an interval of 50 measurements.

							PLANET		EARTH		SUN	
Time /years	**Earth polar angle /rad**	**Planet polar angle /rad from Earth**	**theta1**	**theta2**	**phi1**	**phi2**	**x**	**y**	**x**	**y**	**x**	**y**
0.000	0.000	1.099	0.000	5.526	1.099	1.140	1.437	0.856	1.000	0.000	0.000	0.000
0.038	0.236	1.132	0.236	5.762	1.132	1.274	1.315	0.964	0.972	0.234		
0.075	0.472	1.129	0.472	5.999	1.129	1.403	1.193	1.093	0.890	0.455		
0.113	0.709	1.085	0.709	6.235	1.085	1.526	1.055	1.211	0.759	0.651		
0.150	0.945	1.006	0.945	0.188	1.006	1.641	0.904	1.311	0.586	0.810		
0.188	1.181	0.916	1.181	0.424	0.916	1.744	0.739	1.393	0.380	0.925		
0.226	1.417	0.849	1.417	0.660	0.849	1.832	0.565	1.455	0.153	0.988		
0.263	1.653	0.823	1.653	0.896	0.823	1.897	0.382	1.497	-0.082	0.997		
0.301	1.890	0.841	1.890	1.133	0.841	1.933	0.193	1.516	-0.313	0.950		
0.338	2.126	0.897	2.126	1.369	0.897	1.929	0.001	1.511	-0.527	0.850		
0.376	2.362	0.981	2.362	1.605	0.981	1.884	-0.190	1.481	-0.711	0.703		
0.413	2.598	1.088	2.598	1.841	1.088	1.807	-0.379	1.427	-0.856	0.517		
0.451	2.834	1.212	2.834	2.077	1.212	1.729	-0.561	1.348	-0.953	0.303		
0.489	3.070	1.349	3.070	2.314	1.349	1.681	-0.733	1.244	-0.997	0.071		
0.526	3.307	1.496	3.307	2.550	1.496	1.679	-0.891	1.119	-0.986	-0.164		
0.564	3.543	1.652	3.543	2.786	1.652	1.722	-1.032	0.970	-0.921	-0.391		
0.601	3.779	1.815	3.779	3.022	1.815	1.800	-1.149	0.787	-0.804	-0.595		
0.639	4.015	1.984	4.015	3.258	1.984	1.906	-1.250	0.620	-0.642	-0.767		
0.677	4.251	2.157	4.251	3.494	2.157	2.033	-1.325	0.431	-0.445	-0.896		
0.714	4.488	2.333	4.488	3.731	2.333	2.174	-1.369	0.225	-0.223	-0.975		
0.752	4.724	2.513	4.724	3.967	2.513	2.327	-1.386	0.016	0.011	-1.000		
0.789	4.960	2.694	4.960	4.203	2.694	2.487	-1.373	-0.193	0.245	-0.970		
0.827	5.196	2.877	5.196	4.439	2.877	2.653	-1.331	-0.398	0.465	-0.885		
0.865	5.432	3.060	5.432	4.675	3.060	2.823	-1.264	-0.594	0.659	-0.752		
0.902	5.669	-3.040	5.669	4.912	-3.040	2.995	-1.170	-0.779	0.817	-0.577		
0.940	5.905	-2.858	5.905	5.148	-2.858	-3.114	-1.053	-0.948	0.929	-0.369		
0.977	6.141	-2.676	6.141	5.384	-2.676	-2.939	-0.915	-1.099	0.990	-0.142		
1.015	0.094	-2.496	0.094	5.620	-2.496	-2.765	-0.756	-1.227	0.996	0.094		
1.053	0.330	-2.317	0.330	5.856	-2.317	-2.591	-0.586	-1.333	0.946	0.324		
1.090	0.566	-2.141	0.566	6.093	-2.141	-2.418	-0.408	-1.418	0.844	0.537		
1.128	0.802	-1.966	0.802	0.046	-1.966	-2.246	-0.220	-1.476	0.695	0.719		
1.165	1.039	-1.793	1.039	0.282	-1.793	-2.077	-0.028	-1.507	0.507	0.862		
1.203	1.275	-1.622	1.275	0.518	-1.622	-1.908	0.164	-1.516	0.292	0.957		

This data is generated from the exact Kepler orbits, rather than the actual historical data Kepler obtained from Tycho Brahe!

Tycho Brahe
1546-1601



Planets and Sun not to scale!
Axes scales in AU.

What if the Home planet orbit is <u>not</u> circular?

Let's assume we know Kepler's laws!

$$r=\frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta},\quad x=r\cos\theta,\ y=r\sin\theta$$

$$t=T\left(1-\varepsilon^2\right)^{\frac{3}{2}}\tfrac{1}{2\pi}\int_{\theta_0}^{\theta}\frac{d\lambda}{\left(1-\varepsilon\cos\lambda\right)^2}$$ *From Kepler II,III*

$$T^2=\frac{4\pi^2}{G(M+m)}a^3\approx\frac{4\pi^2}{GM}a^3$$ *Kepler III*

Kepler II: $$\frac{dA}{dt}=\tfrac{1}{2}\sqrt{G(m+M)(1-\varepsilon^2)a}=\tfrac{1}{2}r^2\dot{\theta}$$

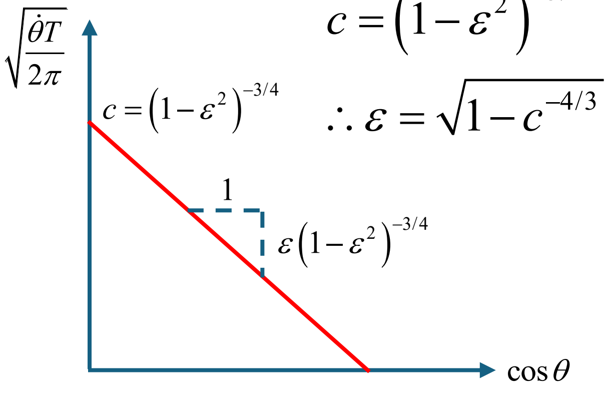
$$\therefore\dot{\theta}=\frac{\sqrt{G(m+M)(1-\varepsilon^2)a}}{a^2\left(1-\varepsilon^2\right)^2}\left(1-\varepsilon\cos\theta\right)^2$$

$$\therefore\dot{\theta}=\sqrt{\frac{G(m+M)}{a^3}}\left(1-\varepsilon^2\right)^{-3/2}\left(1-\varepsilon\cos\theta\right)^2$$

$$\therefore\dot{\theta}=\sqrt{\frac{4\pi^2}{T^2}}\left(1-\varepsilon^2\right)^{-3/2}\left(1-\varepsilon\cos\theta\right)^2$$

$$\therefore\sqrt{\dot{\theta}}=\sqrt{\frac{2\pi}{T}}\left(1-\varepsilon^2\right)^{-3/4}\left(1-\varepsilon\cos\theta\right)$$

$$\therefore\boxed{\sqrt{\frac{\dot{\theta}T}{2\pi}}=\left(1-\varepsilon^2\right)^{-3/4}-\varepsilon\left(1-\varepsilon^2\right)^{-3/4}\cos\theta}$$

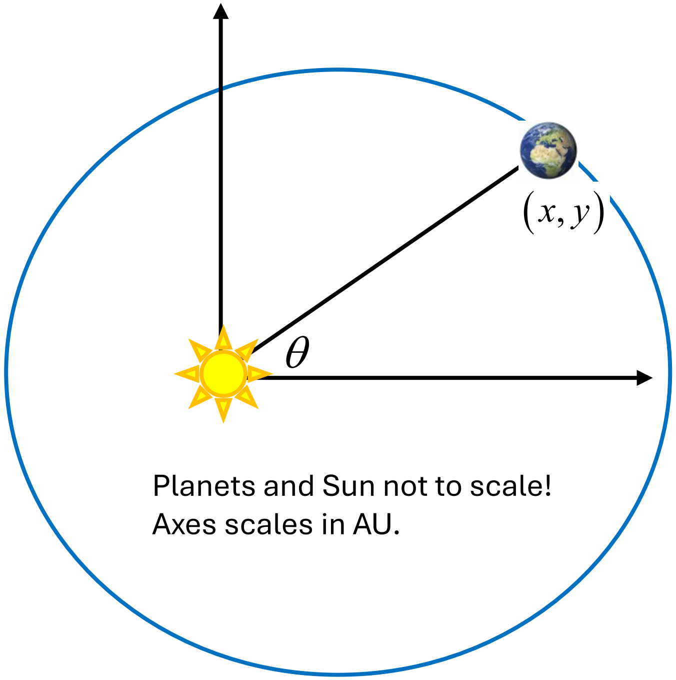


$$c=\left(1-\varepsilon^2\right)^{-3/4}$$

$$\therefore\varepsilon=\sqrt{1-c^{-4/3}}$$

If know T, $\theta(t)$, $d\theta/dt$ then can work out eccentricity ε. Get a from T. Hence calculate $x(t)$, $y(t)$.

What if the Home planet orbit is not circular? (cont.)



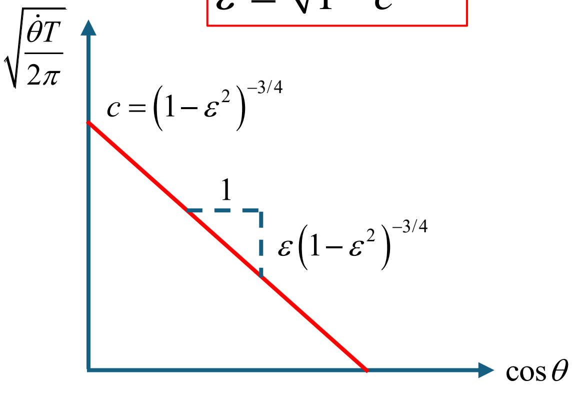
Planets and Sun not to scale!
Axes scales in AU.

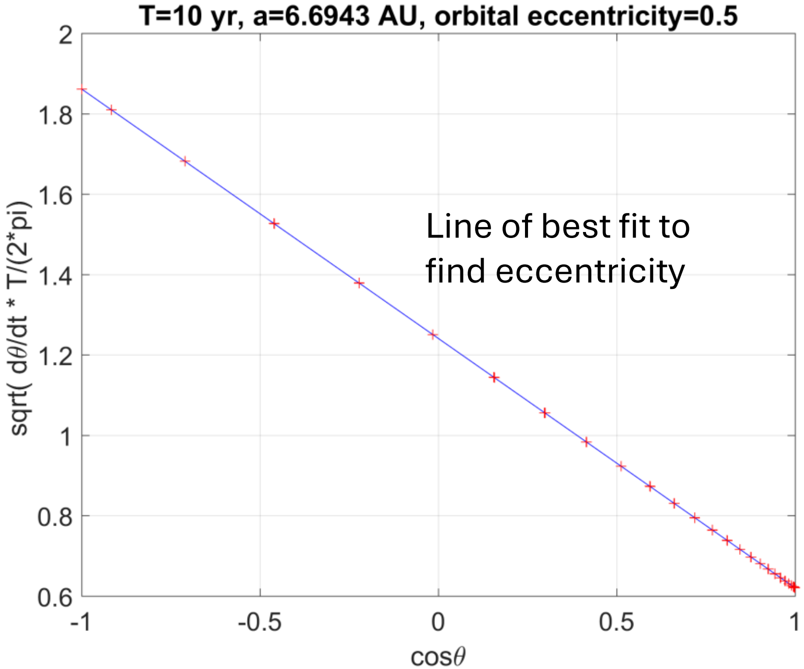
$$\boxed{\begin{aligned} &r=\frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta},\quad x=r\cos\theta,\ y=r\sin\theta \\ &t=T\left(1-\varepsilon^2\right)^{\frac{3}{2}}\tfrac{1}{2\pi}\int_{\theta_0}^{\theta}\frac{d\lambda}{\left(1-\varepsilon\cos\lambda\right)^2} \\ &\left(\frac{a}{\mathrm{AU}}\right)\approx\left(\frac{M}{M_\odot}\right)^{\frac{1}{3}}\left(\frac{T}{\mathrm{Yr}}\right)^{\frac{2}{3}} \quad \textit{Kepler III} \end{aligned}}$$

$$\dot{\theta}=\sqrt{\frac{4\pi^2}{T^2}}\left(1-\varepsilon^2\right)^{-3/2}\left(1-\varepsilon\cos\theta\right)^2$$

$$\therefore\boxed{\sqrt{\frac{\dot{\theta}T}{2\pi}}=\left(1-\varepsilon^2\right)^{-3/4}-\varepsilon\left(1-\varepsilon^2\right)^{-3/4}\cos\theta}$$

$$\boxed{\varepsilon=\sqrt{1-c^{-4/3}}}$$

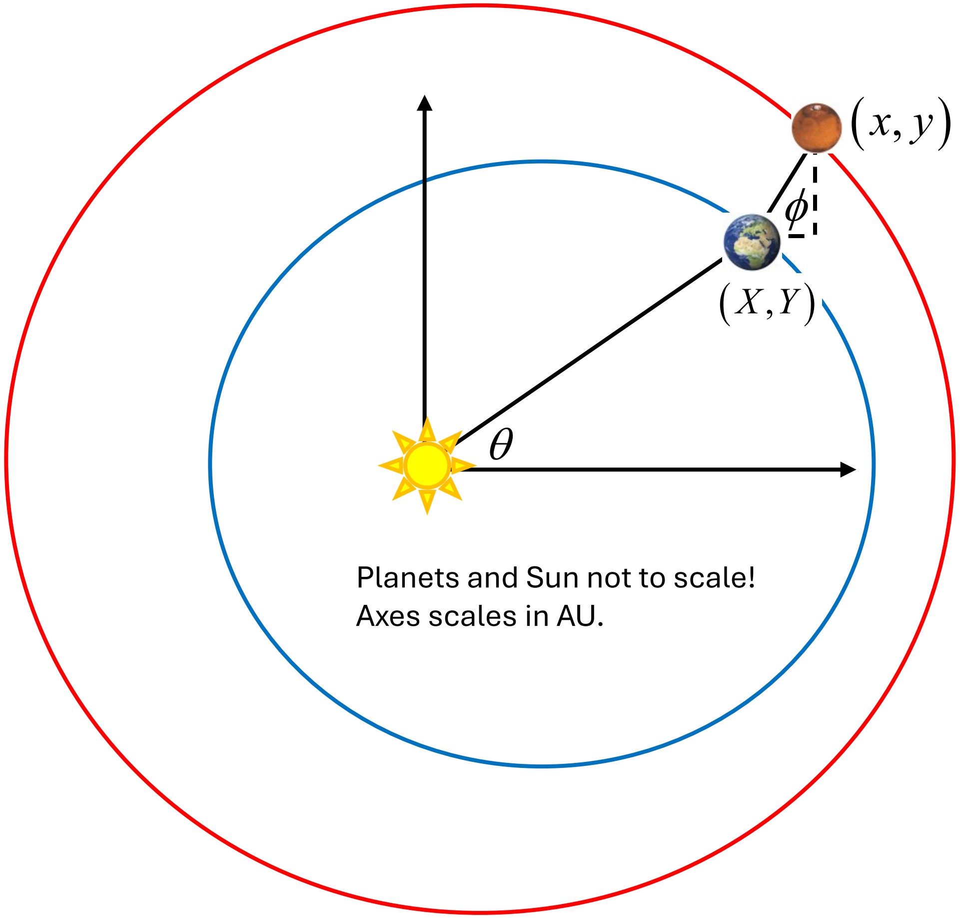




Line of best fit to find eccentricity

If know T, $\theta(t)$, $d\theta/dt$ then can work out eccentricity ε. Get a from T. Hence calculate $x(t)$, $y(t)$.

Calculating planet orbit from elliptical orbit of home planet



Planets and Sun not to scale!
Axes scales in AU.

Exact orbit using Kepler's laws
Mars: $a = 1.523,\ \varepsilon = 0.09$; Earth: $a = 1.00,\ \varepsilon = 0.01$.

Assume we now know what the home planet's orbit is.

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta},\quad X = r\cos\theta,\ Y = r\sin\theta$$

$$t = T_H\left(1-\varepsilon^2\right)^{\frac{3}{2}}\frac{1}{2\pi}\int_{\theta_0}^{\theta}\frac{d\lambda}{\left(1-\varepsilon\cos\lambda\right)^2}$$

$$\left(\frac{a}{\text{AU}}\right) \approx \left(\frac{M}{M_\odot}\right)^{\frac{1}{3}}\left(\frac{T_H}{\text{Yr}}\right)^{\frac{2}{3}}$$

Measure angles θ and ϕ at times separated by one Mars orbital period T. Hence (x,y) coordinates of planet *should remain the same.*

$$\tan\phi = \frac{y-Y}{x-X}$$
$$\therefore y = Y + \left(x-X\right)\tan\phi$$

$$T \approx 1.523^{1.5}\ \text{yr} = 1.88\ \text{yr}\ \ (687\ \text{days})$$

e.g Mars orbital period

$$\therefore y = Y_1 + \left(x-X_1\right)\tan\phi_1$$
$$\therefore y = Y_2 + \left(x-X_2\right)\tan\phi_2$$
$$\therefore Y_1 + \left(x-X_1\right)\tan\phi_1 = Y_2 + \left(x-X_2\right)\tan\phi_2$$
$$\therefore x\left(\tan\phi_2 - \tan\phi_1\right) = Y_1 - Y_2 + X_2\tan\phi_2 - X_1\tan\phi_1$$
$$\therefore x = \frac{Y_1 - Y_2 + X_2\tan\phi_2 - X_1\tan\phi_1}{\tan\phi_2 - \tan\phi_1}$$

$$\therefore x = \frac{Y_1 - Y_2 + X_2\tan\phi_2 - X_1\tan\phi_1}{\tan\phi_2 - \tan\phi_1}$$
$$\therefore y = Y_1 + \left(x-X_1\right)\tan\phi_1$$
$$r_{1,2} = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta_{1,2}},\quad X_{1,2} = r_{1,2}\cos\theta_{1,2},\ Y_{1,2} = r_{1,2}\sin\theta_{1,2}$$

Use this expression to work out orbit of planet using the angles θ *and* ϕ*. Note the latter could have been measured from Earth since ancient times!*

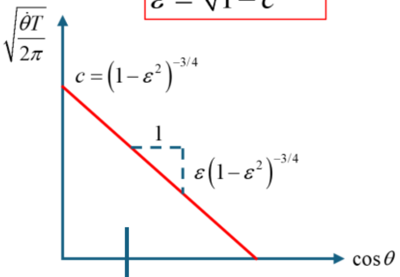
Astronomical Unit (AU) *defined* in 2012

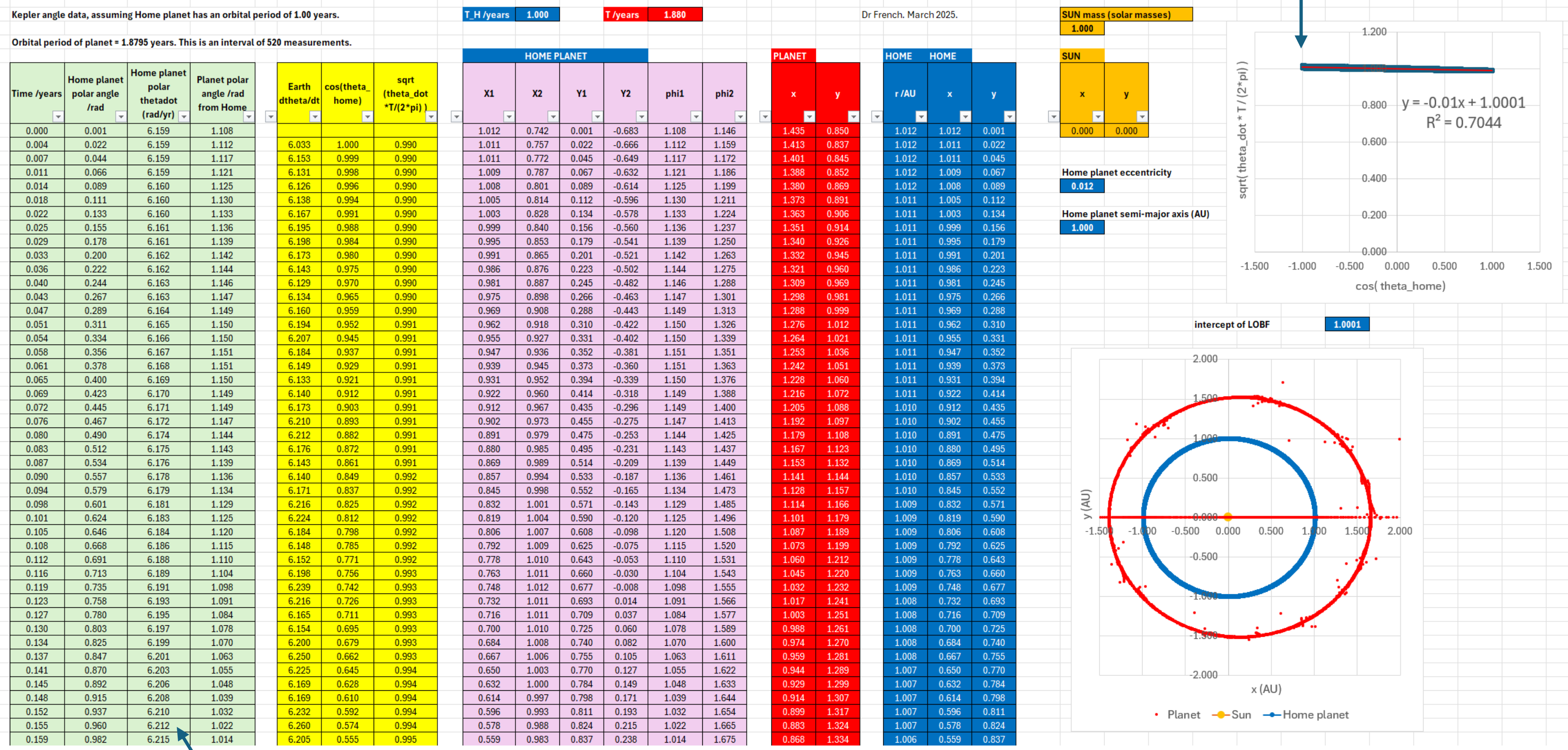
$$\therefore x = \frac{Y_1 - Y_2 + X_2 \tan\phi_2 - X_1 \tan\phi_1}{\tan\phi_2 - \tan\phi_1}$$

$$\therefore y = Y_1 + \left(x - X_1\right)\tan\phi_1$$

$$r_{1,2} = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta_{1,2}}, \quad X_{1,2} = r_{1,2}\cos\theta_{1,2}, \; Y_{1,2} = r_{1,2}\sin\theta_{1,2}$$

$$\boxed{\varepsilon = \sqrt{1-c^{-4/3}}}$$





Kepler angle data, assuming Home planet has an orbital period of 1.00 years.

T_H /years 1.000

T /years 1.880

Dr French. March 2025.

SUN mass (solar masses) 1.000

Orbital period of planet = 1.8795 years. This is an interval of 520 measurements.

							HOME PLANET						PLANET		HOME	HOME	
Time /years	Home planet polar angle /rad	Home planet polar thetadot (rad/yr)	Planet polar angle /rad from Home	Earth dtheta/dt	cos(theta_home)	sqrt (theta_dot *T/(2*pi))	X1	X2	Y1	Y2	phi1	phi2	x	y	r /AU	x	y
0.000	0.001	6.159	1.108				1.012	0.742	0.001	-0.683	1.108	1.146	1.435	0.850	1.012	1.012	0.001
0.004	0.022	6.159	1.112	6.033	1.000	0.990	1.011	0.757	0.022	-0.666	1.112	1.159	1.413	0.837	1.012	1.011	0.022
0.007	0.044	6.159	1.117	6.153	0.999	0.990	1.011	0.772	0.045	-0.649	1.117	1.172	1.401	0.845	1.012	1.011	0.045
0.011	0.066	6.159	1.121	6.131	0.998	0.990	1.009	0.787	0.067	-0.632	1.121	1.186	1.388	0.852	1.012	1.009	0.067
0.014	0.089	6.160	1.125	6.126	0.996	0.990	1.008	0.801	0.089	-0.614	1.125	1.199	1.380	0.869	1.012	1.008	0.089
0.018	0.111	6.160	1.130	6.138	0.994	0.990	1.005	0.814	0.112	-0.596	1.130	1.211	1.373	0.891	1.011	1.005	0.112
0.022	0.133	6.160	1.133	6.167	0.991	0.990	1.003	0.828	0.134	-0.578	1.133	1.224	1.363	0.906	1.011	1.003	0.134
0.025	0.155	6.161	1.136	6.195	0.988	0.990	0.999	0.840	0.156	-0.560	1.136	1.237	1.351	0.914	1.011	0.999	0.156
0.029	0.178	6.161	1.139	6.198	0.984	0.990	0.995	0.853	0.179	-0.541	1.139	1.250	1.340	0.926	1.011	0.995	0.179
0.033	0.200	6.162	1.142	6.173	0.980	0.990	0.991	0.865	0.201	-0.521	1.142	1.263	1.332	0.945	1.011	0.991	0.201
0.036	0.222	6.162	1.144	6.143	0.975	0.990	0.986	0.876	0.223	-0.502	1.144	1.275	1.321	0.960	1.011	0.986	0.223
0.040	0.244	6.163	1.146	6.129	0.970	0.990	0.981	0.887	0.245	-0.482	1.146	1.288	1.309	0.969	1.011	0.981	0.245
0.043	0.267	6.163	1.147	6.134	0.965	0.990	0.975	0.898	0.266	-0.463	1.147	1.301	1.298	0.981	1.011	0.975	0.266
0.047	0.289	6.164	1.149	6.160	0.959	0.990	0.969	0.908	0.288	-0.443	1.149	1.313	1.288	0.999	1.011	0.969	0.288
0.051	0.311	6.165	1.150	6.194	0.952	0.991	0.962	0.918	0.310	-0.422	1.150	1.326	1.276	1.012	1.011	0.962	0.310
0.054	0.334	6.166	1.150	6.207	0.945	0.991	0.955	0.927	0.331	-0.402	1.150	1.339	1.264	1.021	1.011	0.955	0.331
0.058	0.356	6.167	1.151	6.184	0.937	0.991	0.947	0.936	0.352	-0.381	1.151	1.351	1.253	1.036	1.011	0.947	0.352
0.061	0.378	6.168	1.151	6.149	0.929	0.991	0.939	0.945	0.373	-0.360	1.151	1.363	1.242	1.051	1.011	0.939	0.373
0.065	0.400	6.169	1.150	6.133	0.921	0.991	0.931	0.952	0.394	-0.339	1.150	1.376	1.228	1.060	1.011	0.931	0.394
0.069	0.423	6.170	1.149	6.140	0.912	0.991	0.922	0.960	0.414	-0.318	1.149	1.388	1.216	1.072	1.011	0.922	0.414
0.072	0.445	6.171	1.149	6.173	0.903	0.991	0.912	0.967	0.435	-0.296	1.149	1.400	1.205	1.088	1.010	0.912	0.435
0.076	0.467	6.172	1.147	6.210	0.893	0.991	0.902	0.973	0.455	-0.275	1.147	1.413	1.192	1.097	1.010	0.902	0.455
0.080	0.490	6.174	1.144	6.212	0.882	0.991	0.891	0.979	0.475	-0.253	1.144	1.425	1.179	1.108	1.010	0.891	0.475
0.083	0.512	6.175	1.143	6.176	0.872	0.991	0.880	0.985	0.495	-0.231	1.143	1.437	1.167	1.123	1.010	0.880	0.495
0.087	0.534	6.176	1.139	6.143	0.861	0.991	0.869	0.989	0.514	-0.209	1.139	1.449	1.153	1.132	1.010	0.869	0.514
0.090	0.557	6.178	1.136	6.140	0.849	0.992	0.857	0.994	0.533	-0.187	1.136	1.461	1.141	1.144	1.010	0.857	0.533
0.094	0.579	6.179	1.134	6.171	0.837	0.992	0.845	0.998	0.552	-0.165	1.134	1.473	1.128	1.157	1.010	0.845	0.552
0.098	0.601	6.181	1.129	6.216	0.825	0.992	0.832	1.001	0.571	-0.143	1.129	1.485	1.114	1.166	1.009	0.832	0.571
0.101	0.624	6.183	1.125	6.224	0.812	0.992	0.819	1.004	0.590	-0.120	1.125	1.496	1.101	1.179	1.009	0.819	0.590
0.105	0.646	6.184	1.120	6.184	0.798	0.992	0.806	1.007	0.608	-0.098	1.120	1.508	1.087	1.189	1.009	0.806	0.608
0.108	0.668	6.186	1.115	6.148	0.785	0.992	0.792	1.009	0.625	-0.075	1.115	1.520	1.073	1.199	1.009	0.792	0.625
0.112	0.691	6.188	1.110	6.152	0.771	0.992	0.778	1.010	0.643	-0.053	1.110	1.531	1.060	1.212	1.009	0.778	0.643
0.116	0.713	6.189	1.104	6.198	0.756	0.993	0.763	1.011	0.660	-0.030	1.104	1.543	1.045	1.220	1.009	0.763	0.660
0.119	0.735	6.191	1.098	6.239	0.742	0.993	0.748	1.012	0.677	-0.008	1.098	1.555	1.032	1.232	1.009	0.748	0.677
0.123	0.758	6.193	1.091	6.216	0.726	0.993	0.732	1.011	0.693	0.014	1.091	1.566	1.017	1.241	1.008	0.732	0.693
0.127	0.780	6.195	1.084	6.165	0.711	0.993	0.716	1.011	0.709	0.037	1.084	1.577	1.003	1.251	1.008	0.716	0.709
0.130	0.803	6.197	1.078	6.154	0.695	0.993	0.700	1.010	0.725	0.060	1.078	1.589	0.988	1.261	1.008	0.700	0.725
0.134	0.825	6.199	1.070	6.200	0.679	0.993	0.684	1.008	0.740	0.082	1.070	1.600	0.974	1.270	1.008	0.684	0.740
0.137	0.847	6.201	1.063	6.250	0.662	0.993	0.667	1.006	0.755	0.105	1.063	1.611	0.959	1.281	1.008	0.667	0.755
0.141	0.870	6.203	1.055	6.225	0.645	0.994	0.650	1.003	0.770	0.127	1.055	1.622	0.944	1.289	1.007	0.650	0.770
0.145	0.892	6.206	1.048	6.169	0.628	0.994	0.632	1.000	0.784	0.149	1.048	1.633	0.929	1.299	1.007	0.632	0.784
0.148	0.915	6.208	1.039	6.169	0.610	0.994	0.614	0.997	0.798	0.171	1.039	1.644	0.914	1.307	1.007	0.614	0.798
0.152	0.937	6.210	1.032	6.232	0.592	0.994	0.596	0.993	0.811	0.193	1.032	1.654	0.899	1.317	1.007	0.596	0.811
0.155	0.960	6.212	1.022	6.260	0.574	0.994	0.578	0.988	0.824	0.215	1.022	1.665	0.883	1.324	1.007	0.578	0.824
0.159	0.982	6.215	1.014	6.205	0.555	0.995	0.559	0.983	0.837	0.238	1.014	1.675	0.868	1.334	1.006	0.559	0.837

SUN	
x	y
0.000	0.000

Home planet eccentricity 0.012

Home planet semi-major axis (AU) 1.000

intercept of LOBF 1.0001

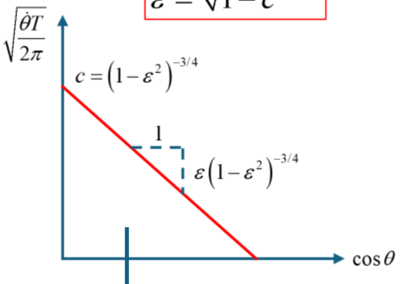
This data is generated from the exact Kepler orbits, rather than the actual historical data Kepler obtained from Tycho Brahe!

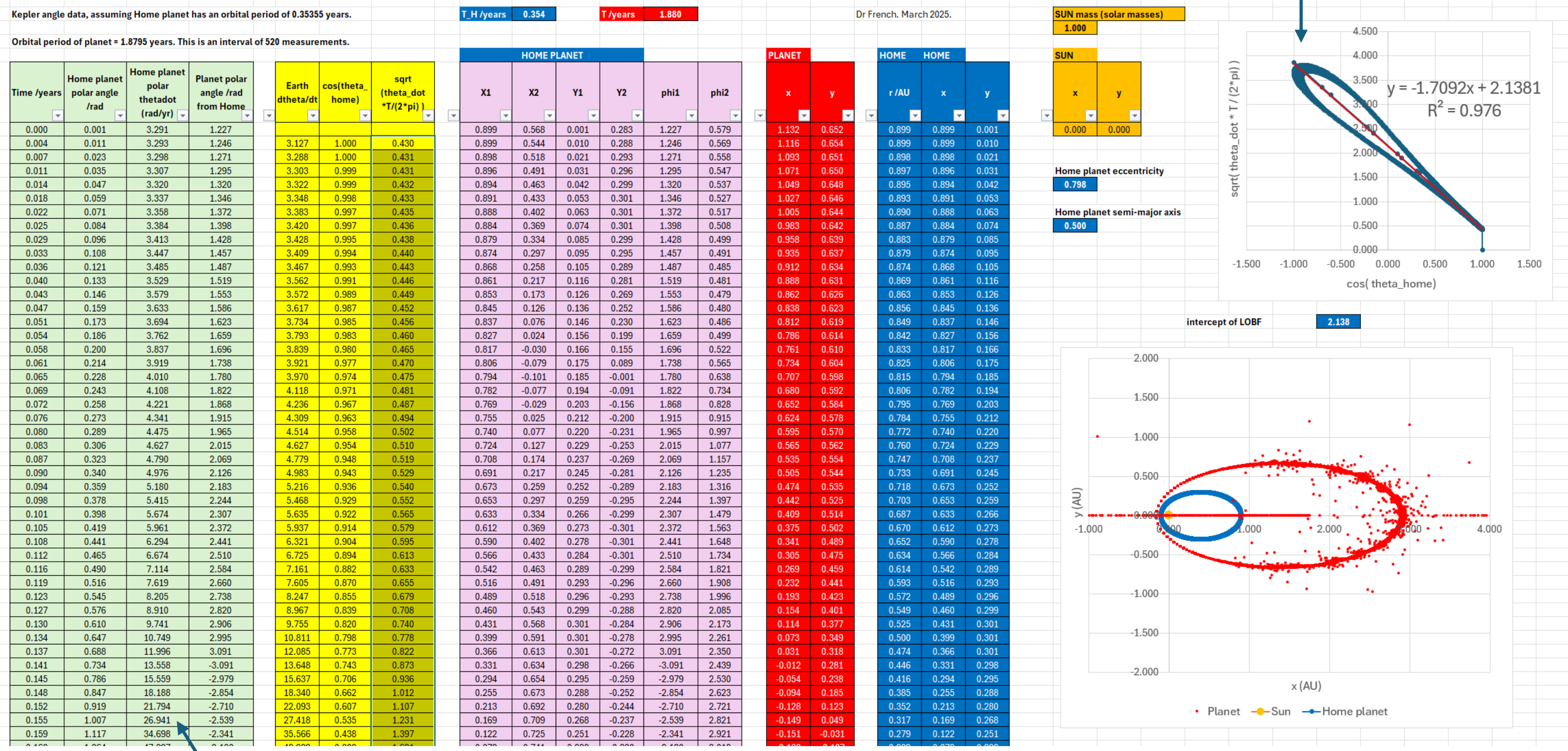
$$\therefore x = \frac{Y_1 - Y_2 + X_2 \tan\phi_2 - X_1 \tan\phi_1}{\tan\phi_2 - \tan\phi_1}$$

$$\therefore y = Y_1 + \left(x - X_1\right)\tan\phi_1$$

$$r_{1,2} = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta_{1,2}}, \quad X_{1,2} = r_{1,2}\cos\theta_{1,2}, \; Y_{1,2} = r_{1,2}\sin\theta_{1,2}$$

$$\varepsilon = \sqrt{1-c^{-4/3}}$$





Kepler angle data, assuming Home planet has an orbital period of 0.35355 years.

Orbital period of planet = 1.8795 years. This is an interval of 520 measurements.

T_H /years	**0.354**
T /years	**1.880**

Dr French. March 2025.

SUN mass (solar masses)	
1.000	
SUN	
x	**y**
0.000	0.000

Home planet eccentricity

0.798

Home planet semi-major axis

0.500

intercept of LOBF **2.138**

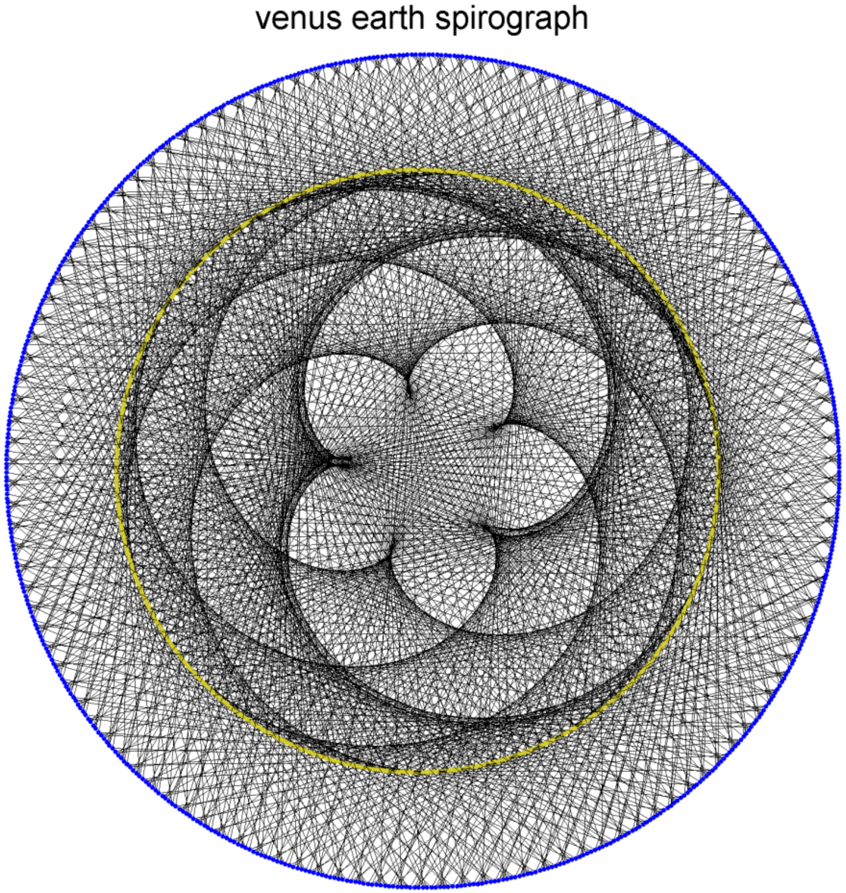
								HOME PLANET						**PLANET**		**HOME**	**HOME**	
Time /years	**Home planet polar angle /rad**	**Home planet polar thetadot (rad/yr)**	**Planet polar angle /rad from Home**	**Earth dtheta/dt**	**cos(theta_ home)**	**sqrt (theta_dot *T/(2*pi))**	**X1**	**X2**	**Y1**	**Y2**	**phi1**	**phi2**	**x**	**y**	**r /AU**	**x**	**y**
0.000	0.001	3.291	1.227				0.899	0.568	0.001	0.283	1.227	0.579	1.132	0.652	0.899	0.899	0.001
0.004	0.011	3.293	1.246	3.127	1.000	0.430	0.899	0.544	0.010	0.288	1.246	0.569	1.116	0.654	0.899	0.899	0.010
0.007	0.023	3.298	1.271	3.288	1.000	0.431	0.898	0.518	0.021	0.293	1.271	0.558	1.093	0.651	0.898	0.898	0.021
0.011	0.035	3.307	1.295	3.303	0.999	0.431	0.896	0.491	0.031	0.296	1.295	0.547	1.071	0.650	0.897	0.896	0.031
0.014	0.047	3.320	1.320	3.322	0.999	0.432	0.894	0.463	0.042	0.299	1.320	0.537	1.049	0.648	0.895	0.894	0.042
0.018	0.059	3.337	1.346	3.348	0.998	0.433	0.891	0.433	0.053	0.301	1.346	0.527	1.027	0.646	0.893	0.891	0.053
0.022	0.071	3.358	1.372	3.383	0.997	0.435	0.888	0.402	0.063	0.301	1.372	0.517	1.005	0.644	0.890	0.888	0.063
0.025	0.084	3.384	1.398	3.420	0.997	0.436	0.884	0.369	0.074	0.301	1.398	0.508	0.983	0.642	0.887	0.884	0.074
0.029	0.096	3.413	1.428	3.428	0.995	0.438	0.879	0.334	0.085	0.299	1.428	0.499	0.958	0.639	0.883	0.879	0.085
0.033	0.108	3.447	1.457	3.409	0.994	0.440	0.874	0.297	0.095	0.295	1.457	0.491	0.935	0.637	0.879	0.874	0.095
0.036	0.121	3.485	1.487	3.467	0.993	0.443	0.868	0.258	0.105	0.289	1.487	0.485	0.912	0.634	0.874	0.868	0.105
0.040	0.133	3.529	1.519	3.562	0.991	0.446	0.861	0.217	0.116	0.281	1.519	0.481	0.888	0.631	0.869	0.861	0.116
0.043	0.146	3.579	1.553	3.572	0.989	0.449	0.853	0.173	0.126	0.269	1.553	0.479	0.862	0.626	0.863	0.853	0.126
0.047	0.159	3.633	1.586	3.617	0.987	0.452	0.845	0.126	0.136	0.252	1.586	0.480	0.838	0.623	0.856	0.845	0.136
0.051	0.173	3.694	1.623	3.734	0.985	0.456	0.837	0.076	0.146	0.230	1.623	0.486	0.812	0.619	0.849	0.837	0.146
0.054	0.186	3.762	1.659	3.793	0.983	0.460	0.827	0.024	0.156	0.199	1.659	0.499	0.786	0.614	0.842	0.827	0.156
0.058	0.200	3.837	1.696	3.839	0.980	0.465	0.817	-0.030	0.166	0.155	1.696	0.522	0.761	0.610	0.833	0.817	0.166
0.061	0.214	3.919	1.738	3.921	0.977	0.470	0.806	-0.079	0.175	0.089	1.738	0.565	0.734	0.604	0.825	0.806	0.175
0.065	0.228	4.010	1.780	3.970	0.974	0.475	0.794	-0.101	0.185	-0.001	1.780	0.638	0.707	0.598	0.815	0.794	0.185
0.069	0.243	4.108	1.822	4.118	0.971	0.481	0.782	-0.077	0.194	-0.091	1.822	0.734	0.680	0.592	0.806	0.782	0.194
0.072	0.258	4.221	1.868	4.236	0.967	0.487	0.769	-0.029	0.203	-0.156	1.868	0.828	0.652	0.584	0.795	0.769	0.203
0.076	0.273	4.341	1.915	4.309	0.963	0.494	0.755	0.025	0.212	-0.200	1.915	0.915	0.624	0.578	0.784	0.755	0.212
0.080	0.289	4.475	1.965	4.514	0.958	0.502	0.740	0.077	0.220	-0.231	1.965	0.997	0.595	0.570	0.772	0.740	0.220
0.083	0.306	4.627	2.015	4.627	0.954	0.510	0.724	0.127	0.229	-0.253	2.015	1.077	0.565	0.562	0.760	0.724	0.229
0.087	0.323	4.790	2.069	4.779	0.948	0.519	0.708	0.174	0.237	-0.269	2.069	1.157	0.535	0.554	0.747	0.708	0.237
0.090	0.340	4.976	2.126	4.983	0.943	0.529	0.691	0.217	0.245	-0.281	2.126	1.235	0.505	0.544	0.733	0.691	0.245
0.094	0.359	5.180	2.183	5.216	0.936	0.540	0.673	0.259	0.252	-0.289	2.183	1.316	0.474	0.535	0.718	0.673	0.252
0.098	0.378	5.415	2.244	5.468	0.929	0.552	0.653	0.297	0.259	-0.295	2.244	1.397	0.442	0.525	0.703	0.653	0.259
0.101	0.398	5.674	2.307	5.635	0.922	0.565	0.633	0.334	0.266	-0.299	2.307	1.479	0.409	0.514	0.687	0.633	0.266
0.105	0.419	5.961	2.372	5.937	0.914	0.579	0.612	0.369	0.273	-0.301	2.372	1.563	0.375	0.502	0.670	0.612	0.273
0.108	0.441	6.294	2.441	6.321	0.904	0.595	0.590	0.402	0.278	-0.301	2.441	1.648	0.341	0.489	0.652	0.590	0.278
0.112	0.465	6.674	2.510	6.725	0.894	0.613	0.566	0.433	0.284	-0.301	2.510	1.734	0.305	0.475	0.634	0.566	0.284
0.116	0.490	7.114	2.584	7.161	0.882	0.633	0.542	0.463	0.289	-0.299	2.584	1.821	0.269	0.459	0.614	0.542	0.289
0.119	0.516	7.619	2.660	7.605	0.870	0.655	0.516	0.491	0.293	-0.296	2.660	1.908	0.232	0.441	0.593	0.516	0.293
0.123	0.545	8.205	2.738	8.247	0.855	0.679	0.489	0.518	0.296	-0.293	2.738	1.996	0.193	0.423	0.572	0.489	0.296
0.127	0.576	8.910	2.820	8.967	0.839	0.708	0.460	0.543	0.299	-0.288	2.820	2.085	0.154	0.401	0.549	0.460	0.299
0.130	0.610	9.741	2.906	9.755	0.820	0.740	0.431	0.568	0.301	-0.284	2.906	2.173	0.114	0.377	0.525	0.431	0.301
0.134	0.647	10.749	2.995	10.811	0.798	0.778	0.399	0.591	0.301	-0.278	2.995	2.261	0.073	0.349	0.500	0.399	0.301
0.137	0.688	11.996	3.091	12.085	0.773	0.822	0.366	0.613	0.301	-0.272	3.091	2.350	0.031	0.318	0.474	0.366	0.301
0.141	0.734	13.558	-3.091	13.648	0.743	0.873	0.331	0.634	0.298	-0.266	-3.091	2.439	-0.012	0.281	0.446	0.331	0.298
0.145	0.786	15.559	-2.979	15.637	0.706	0.936	0.294	0.654	0.295	-0.259	-2.979	2.530	-0.054	0.238	0.416	0.294	0.295
0.148	0.847	18.188	-2.854	18.340	0.662	1.012	0.255	0.673	0.288	-0.252	-2.854	2.623	-0.094	0.185	0.385	0.255	0.288
0.152	0.919	21.794	-2.710	22.093	0.607	1.107	0.213	0.692	0.280	-0.244	-2.710	2.721	-0.128	0.123	0.352	0.213	0.280
0.155	1.007	26.941	-2.539	27.418	0.535	1.231	0.169	0.709	0.268	-0.237	-2.539	2.821	-0.149	0.049	0.317	0.169	0.268
0.159	1.117	34.698	-2.341	35.566	0.438	1.397	0.122	0.725	0.251	-0.228	-2.341	2.921	-0.151	-0.031	0.279	0.122	0.251

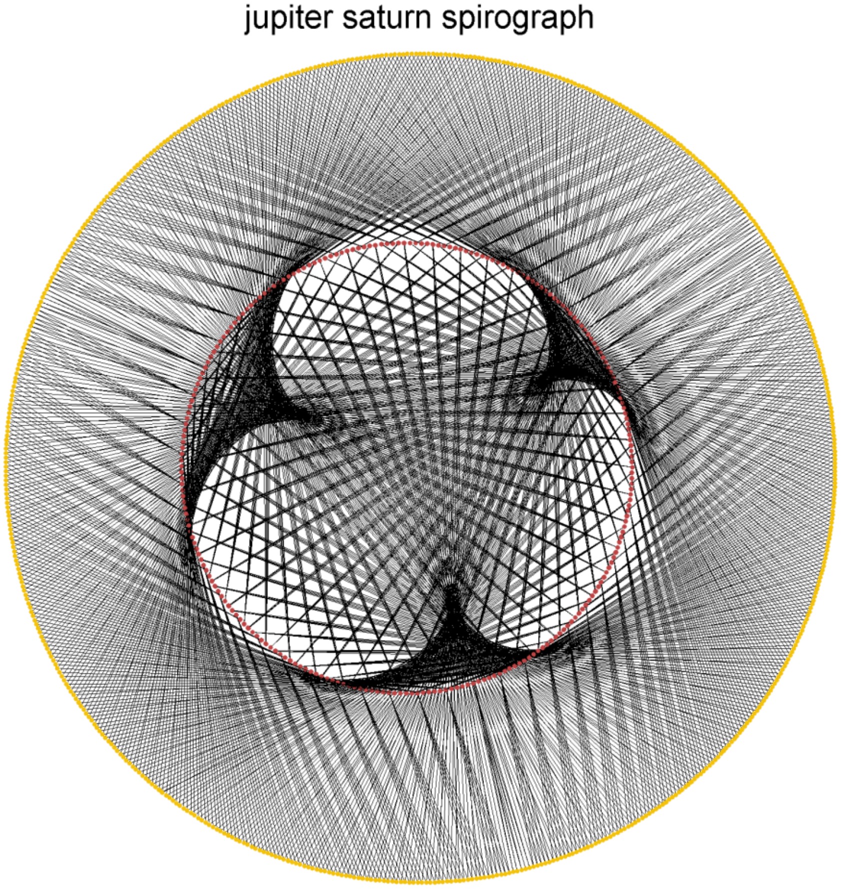
Data generated for a home planet of eccentricity 0.8 and planet of eccentricity 0.9

Now we have a model of $x(t)$ and $y(t)$ of solar system orbits .....
... Construct a solar system spirograph!

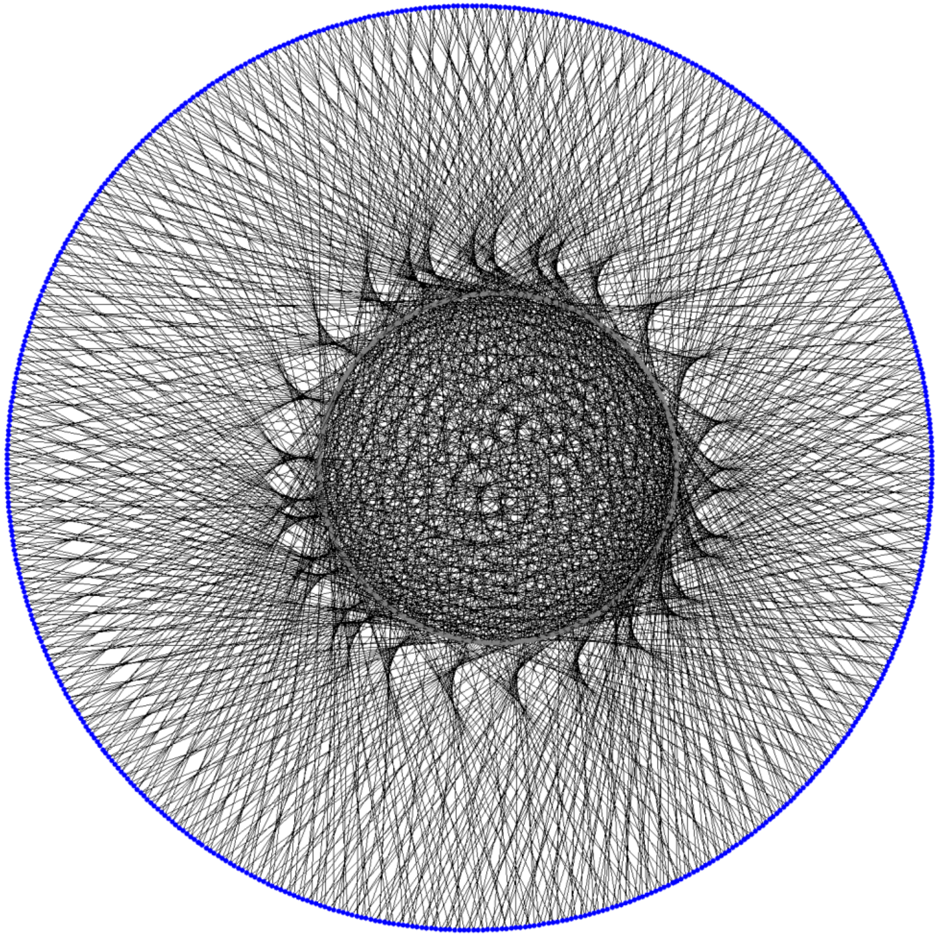
inspired by: https://engaging-data.com/planetary-spirograph

Choose a pair of planets and determine their orbits vs time. At time intervals of Δt, draw a line between the planets and plot this line. Keep going for N orbits of the outermost planet.
$N = $ **10 ,** Δt **=** N x **maximum orbital period /1234**, might be sensible parameters.

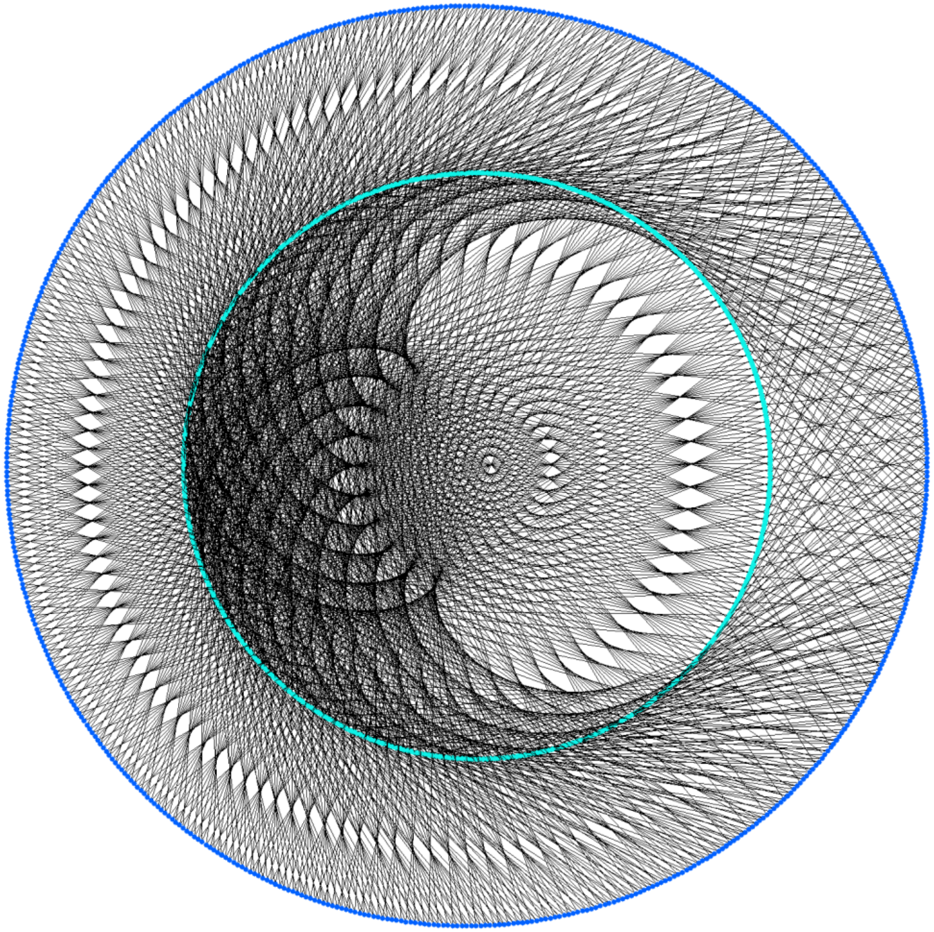

venus earth spirograph


jupiter saturn spirograph

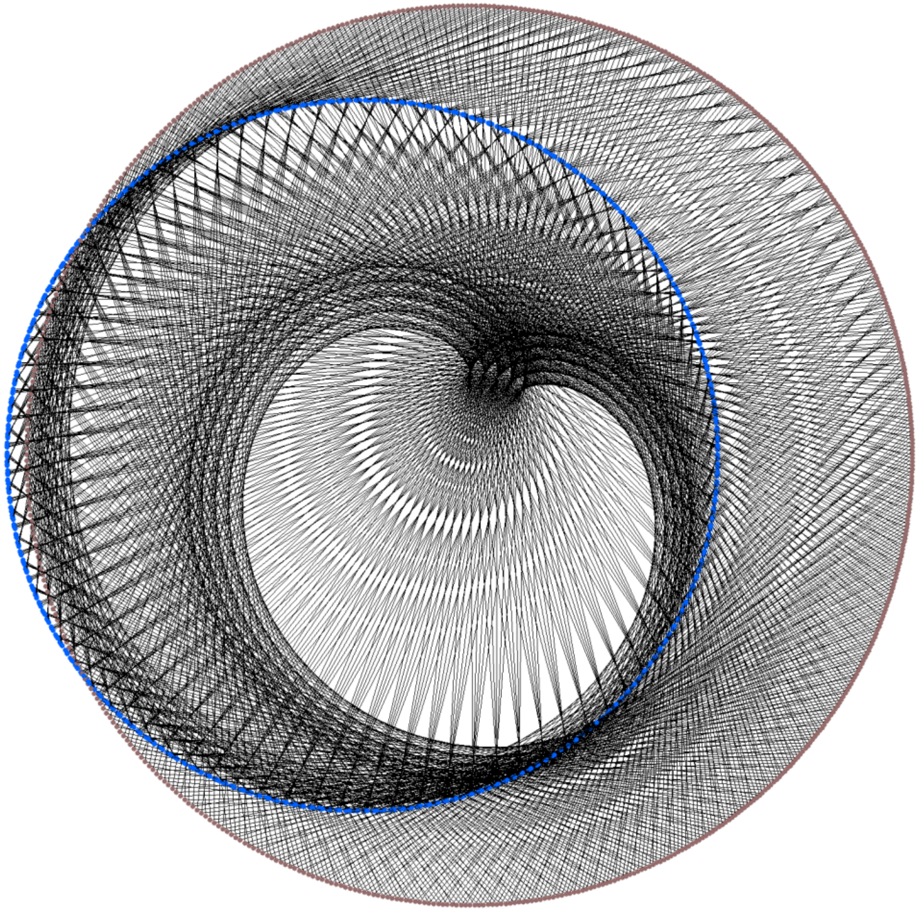
mercury earth spirograph



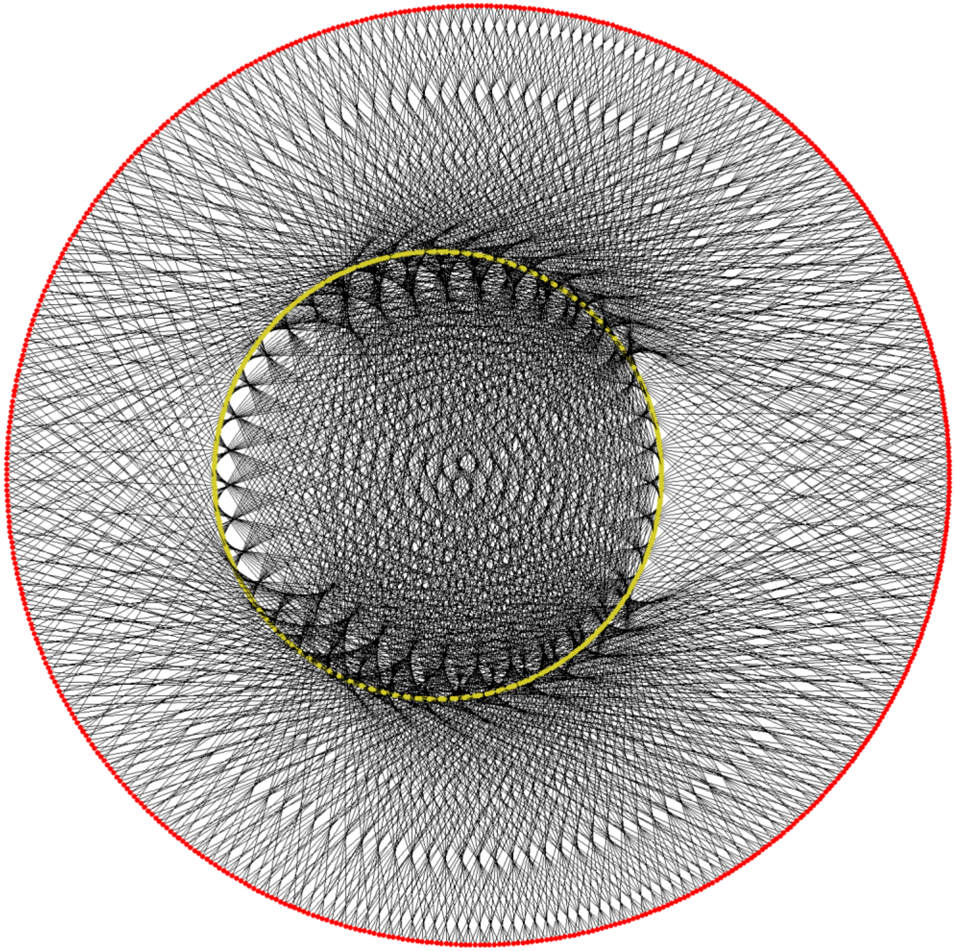
uranus neptune spirograph



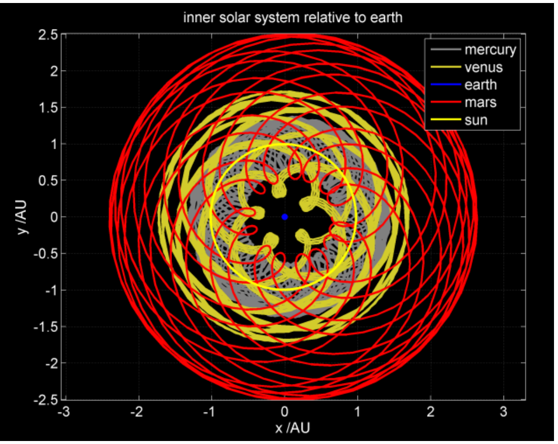
neptune pluto spirograph

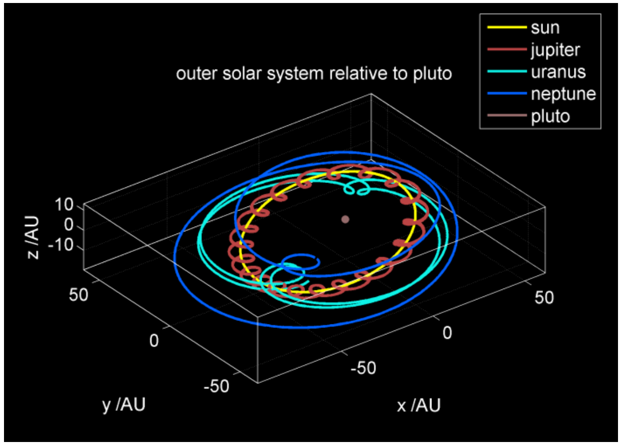


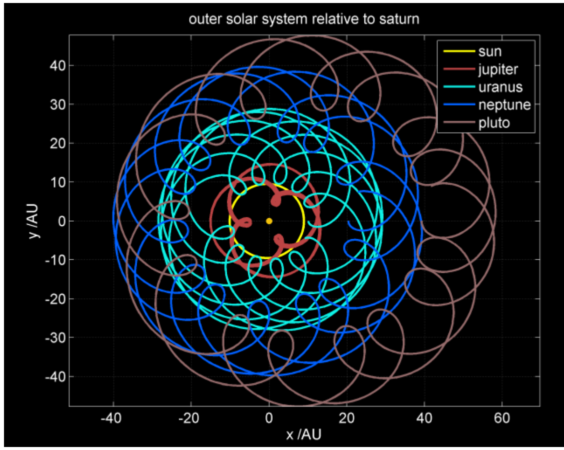
venus mars spirograph

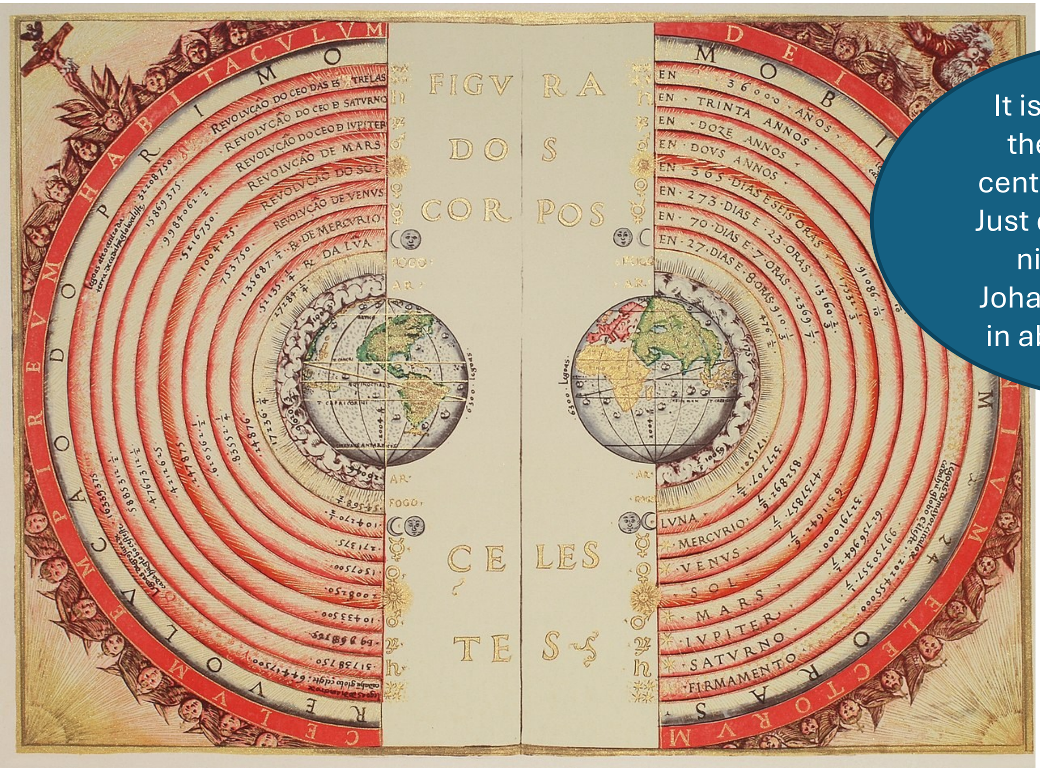


Plot the orbits of the other bodies in the solar system, with a chosen object (e.g. Earth) at a *fixed position at the origin of a Cartesian coordinate system*. i.e. choose a coordinate system where your chosen object is at (0,0,0).



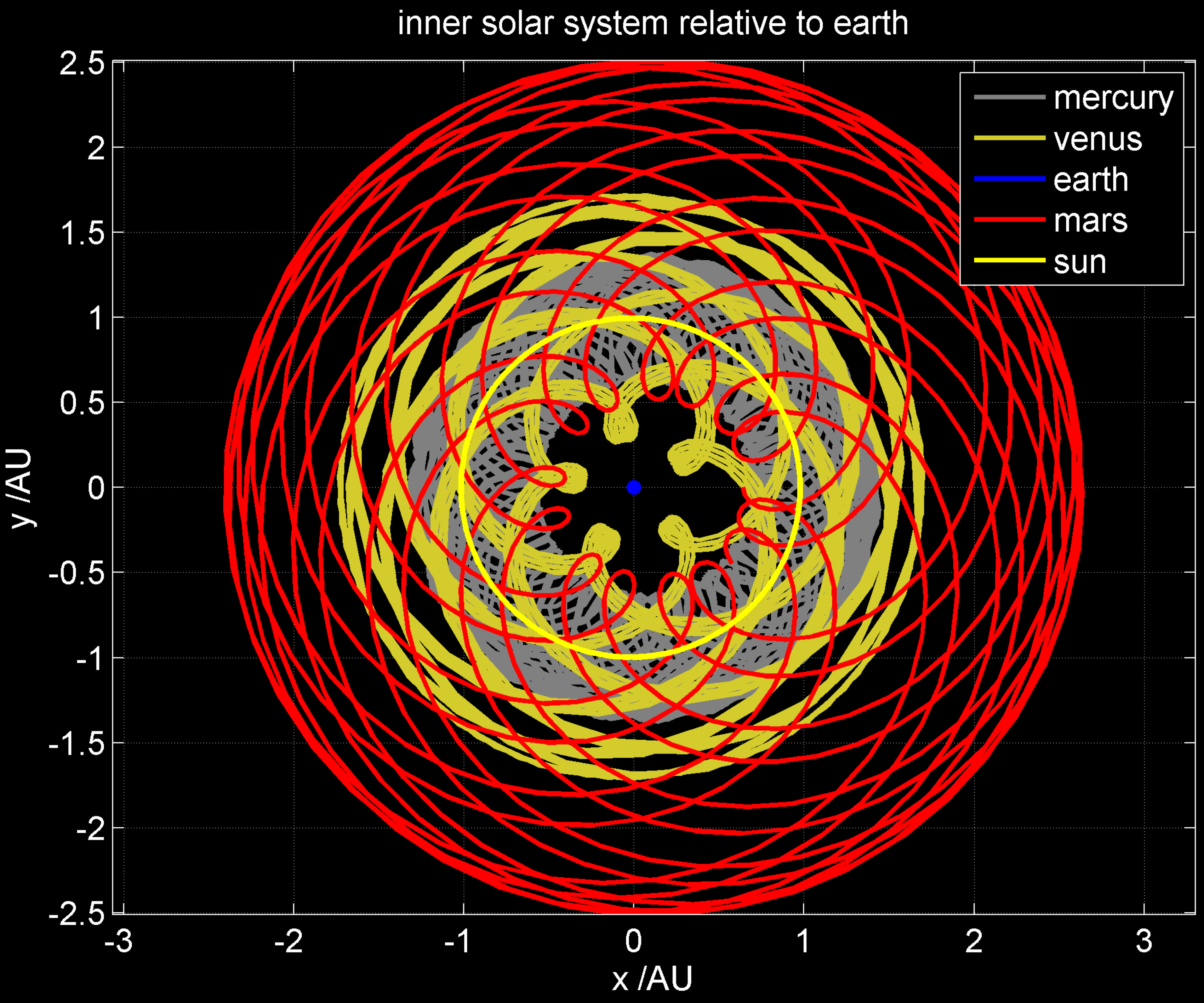


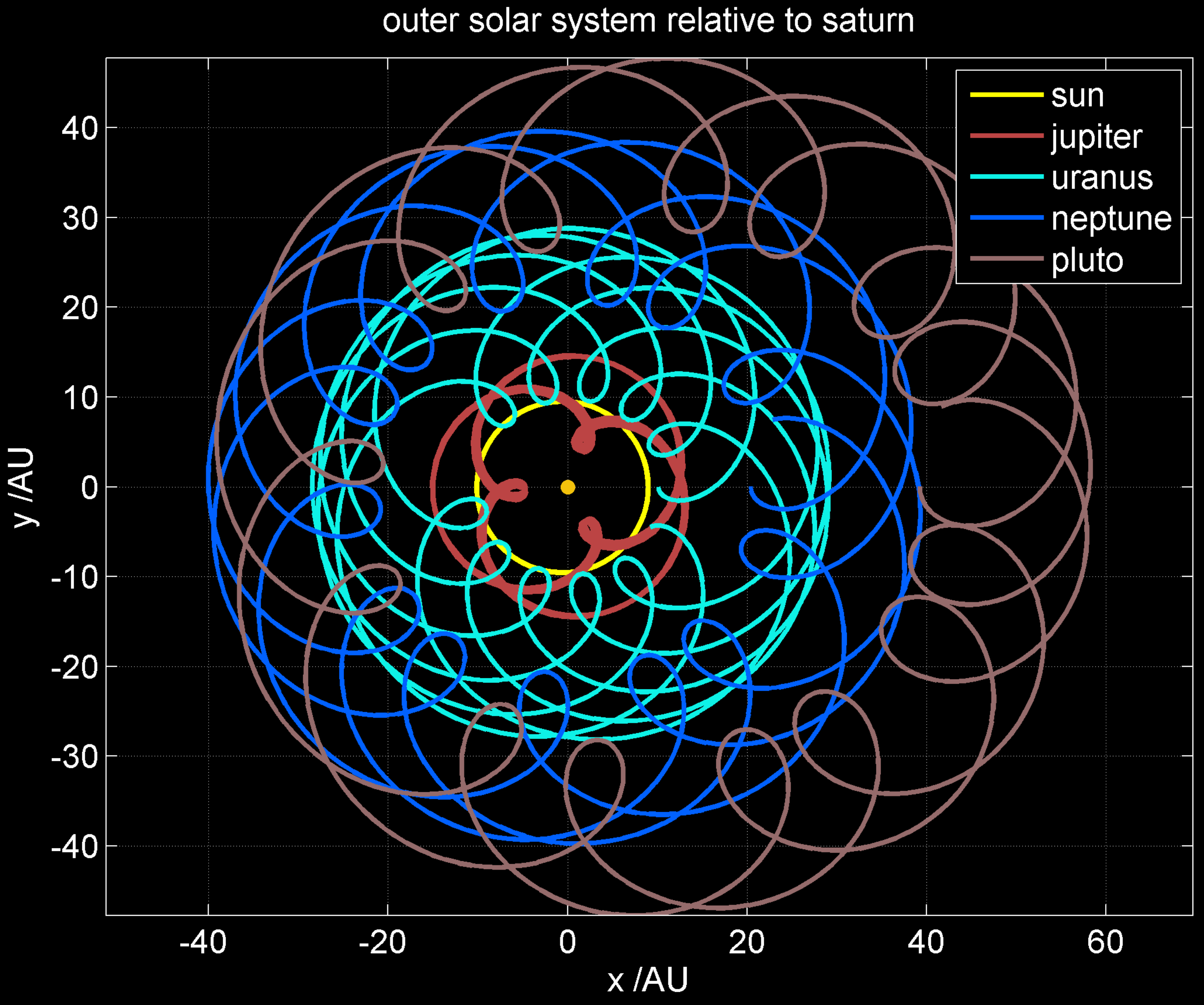




It is perfectly fine for the Earth to be the centre of the Universe! Just don't expect those nice ellipses that Johannes will discover in about 1500 years...

Claudius Ptolemy (100-170 AD)


inner solar system relative to earth
mercury
venus
earth
mars
sun
y /AU
x /AU
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2
-1
0
1
2
3


outer solar system relative to saturn
sun
jupiter
uranus
neptune
pluto
y /AU
x /AU
40
30
20
10
0
-10
-20
-30
-40
-40
-20
0
20
40
60

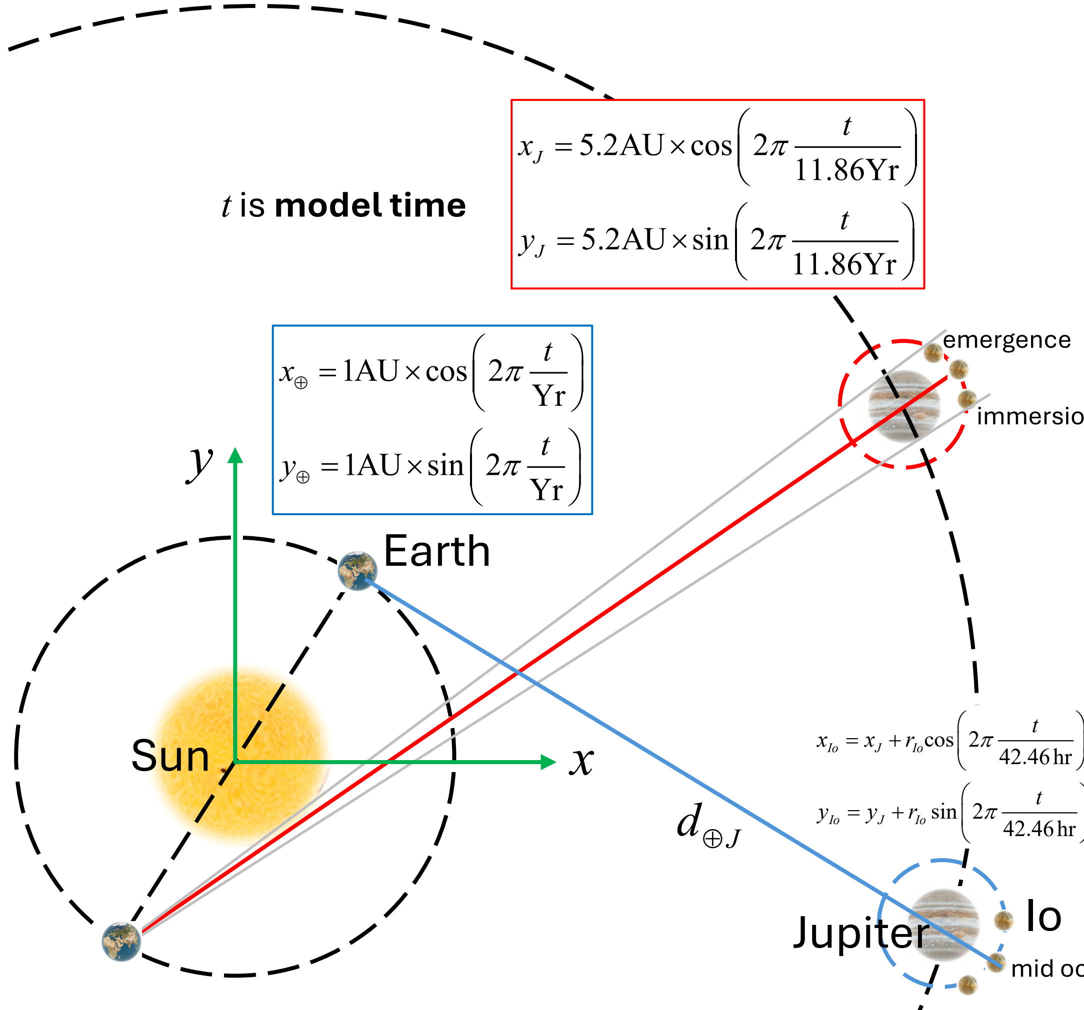
5. Calculating the speed of light using the occultation of Io by Jupiter

r_{Io}

- **Io** orbital radius **421,000km** about Jupiter, period **42.46 hours** (1.769 days)
- **Jupiter** orbital radius about Sun is **5.20AU**, period **11.86 years**
- **Earth** orbital radius about Sun is **1AU**, period **1 year**

Ole Rømer
1644-1710

Christiaan
Huygens
1629-1695



t is **model time**

$$x_J = 5.2\text{AU} \times \cos\left(2\pi \frac{t}{11.86\text{Yr}}\right)$$

$$y_J = 5.2\text{AU} \times \sin\left(2\pi \frac{t}{11.86\text{Yr}}\right)$$

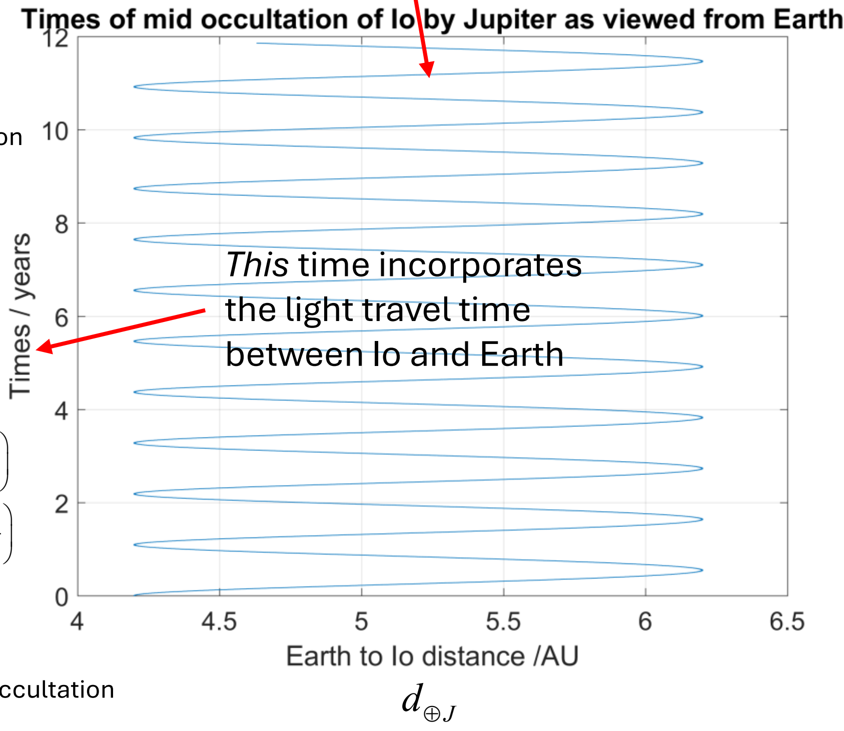
$$x_{\oplus} = 1\text{AU} \times \cos\left(2\pi \frac{t}{\text{Yr}}\right)$$

$$y_{\oplus} = 1\text{AU} \times \sin\left(2\pi \frac{t}{\text{Yr}}\right)$$

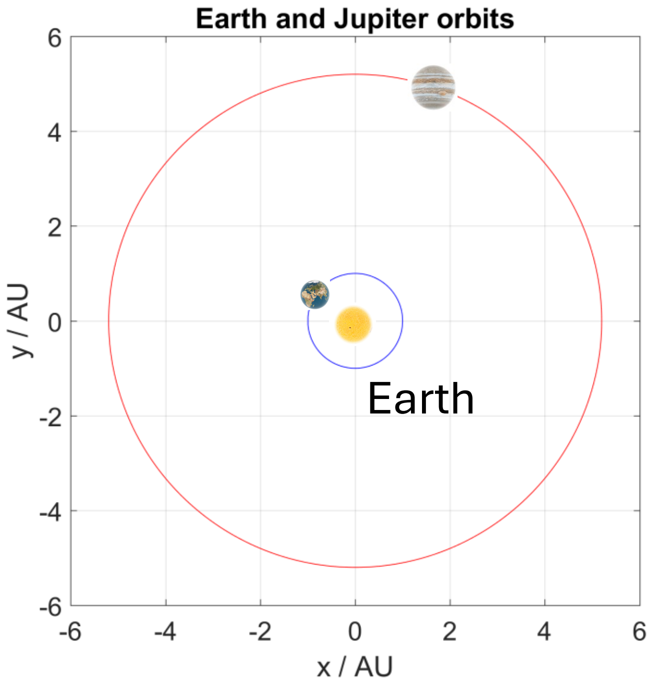
$$x_{Io} = x_J + r_{Io}\cos\left(2\pi \frac{t}{42.46\,\text{hr}}\right)$$

$$y_{Io} = y_J + r_{Io}\sin\left(2\pi \frac{t}{42.46\,\text{hr}}\right)$$

Jupiter orbits more slowly than Earth, so distance between Earth and Jupiter varies

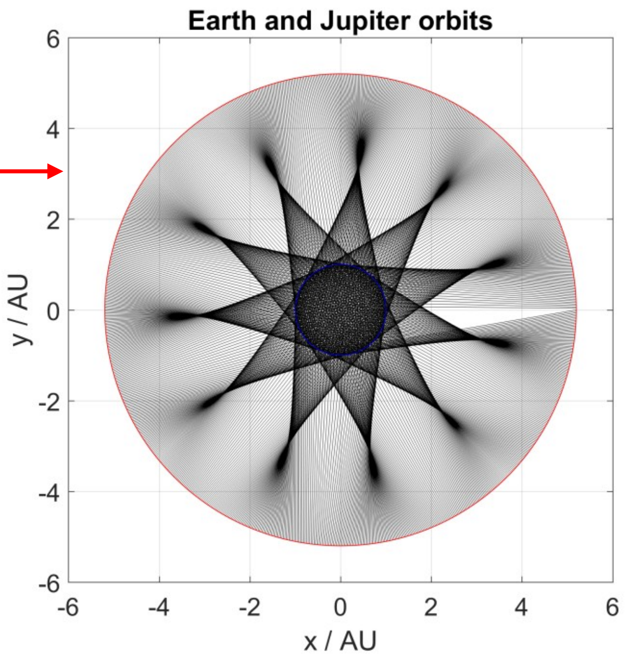


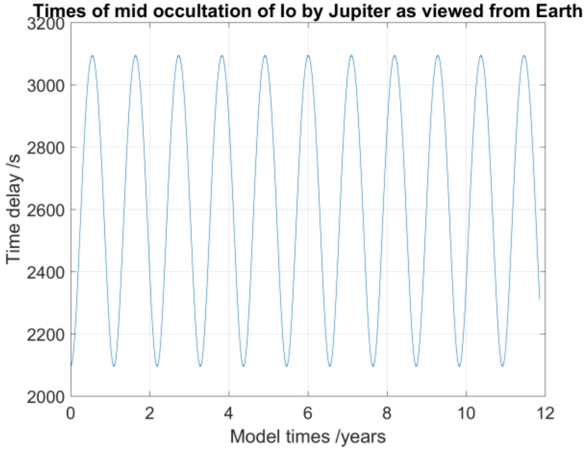
This time incorporates the light travel time between Io and Earth

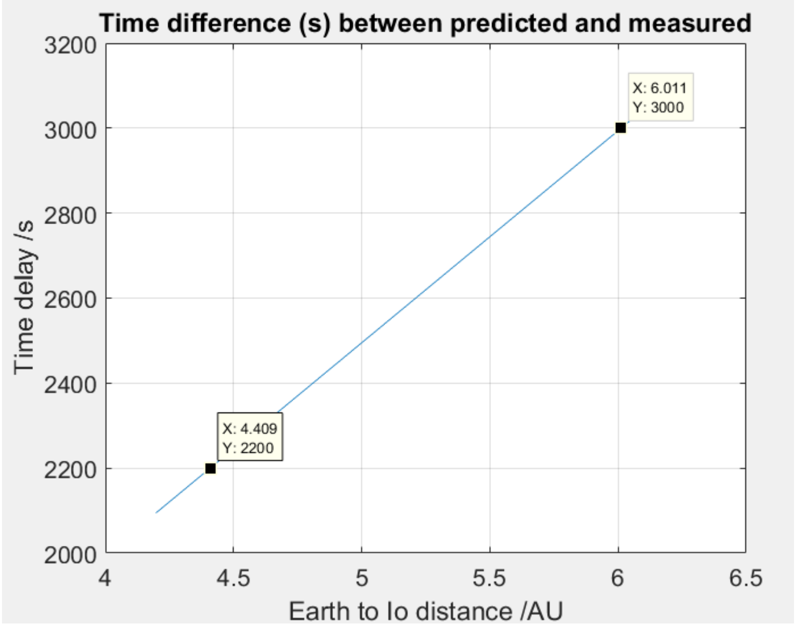


Planets and Sun not to scale!

Line between Earth and Jupiter drawn at 1000 points over one Jupiter orbit





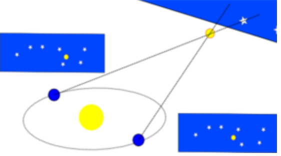


Using Kepler's laws to determine orbital positions vs time of Earth, Jupiter and Io, plot the **time delay** between the **measured** time of a mid-occultation, and *that predicted by the model*, vs Earth to Io distance in AU.

i.e. not accounting for light-travel time delays

Gradient of the Earth to Io distance vs time delay graph is the **speed of light**

$$c = \frac{(6.011 - 4.409) \times 1.496 \times 10^{11}\,\mathrm{m}}{(3000 - 2200)\,\mathrm{s}} = \boxed{3.0 \times 10^{8}\,\mathrm{ms}^{-1}}$$

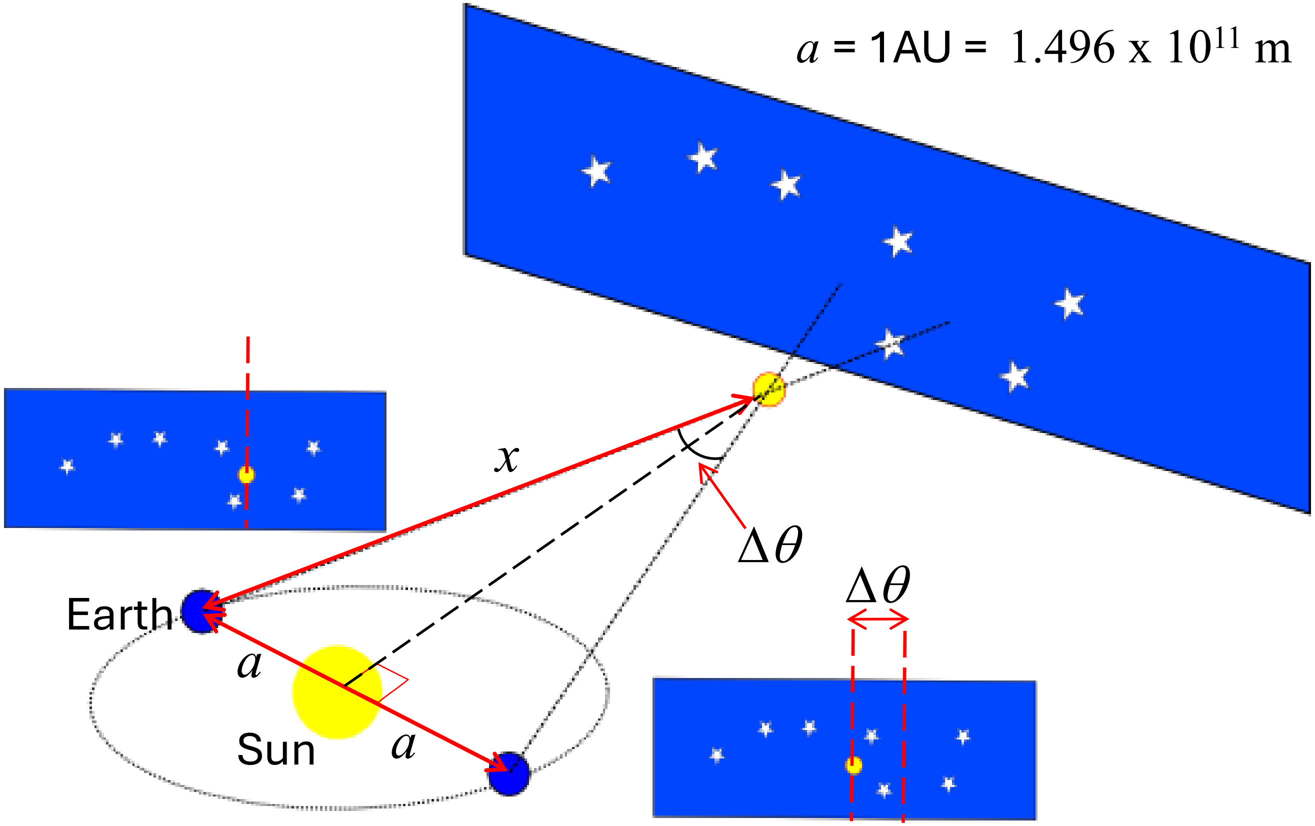


6. Calculating the distance to nearby stars via parallax

Caution! Parallax is often stated as $\Delta\theta/2$

Record the angular change $\Delta\theta$ in the position of a star over the course of a year, i.e. as the Earth orbits the Sun.

This assumes the stars are fixed relative to the Earth over this timescale!



a = 1AU = 1.496×10^{11} m

$$x \sin\tfrac{1}{2}\Delta\theta = a$$

$$\boxed{x = \frac{a}{\sin\frac{1}{2}\Delta\theta}}$$

The parallax of our nearest star outside of the solar system (Proxima Centauri) is $\Delta\theta = 1.53626$ arc-seconds.

$$\Delta\theta = \frac{1.53626^{\circ}}{3600} \qquad \therefore x = \frac{1}{\sin\frac{1}{2}\Delta\theta} = \boxed{268{,}532\,\mathrm{AU}}$$

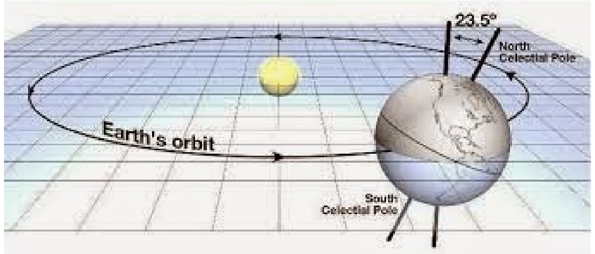
$$x = \boxed{4.02 \times 10^{16}\,\mathrm{m}}$$

$$x = \frac{4.02 \times 10^{16}}{9.461 \times 10^{15}} = \boxed{4.25\ \text{light-years}}$$

Astronomical length scales

Astronomical Unit (approximately the Earth-Sun distance)

$$\boxed{1\text{AU} = 1.496\times10^{11}\,\text{m}}$$



Light-year

$$c = 2.998\times10^{8}\,\text{ms}^{-1}$$

$$t_{year} \approx 365\times24\times3600 = 3.15\times10^{7}\,\text{s}$$

$$t_{year} \approx \pi\times10^{7}\,\text{s}$$

$$1\text{ly} = ct_{year} = \boxed{9.461\times10^{15}\,\text{m}}$$

calculated from more precise light speeds and year durations

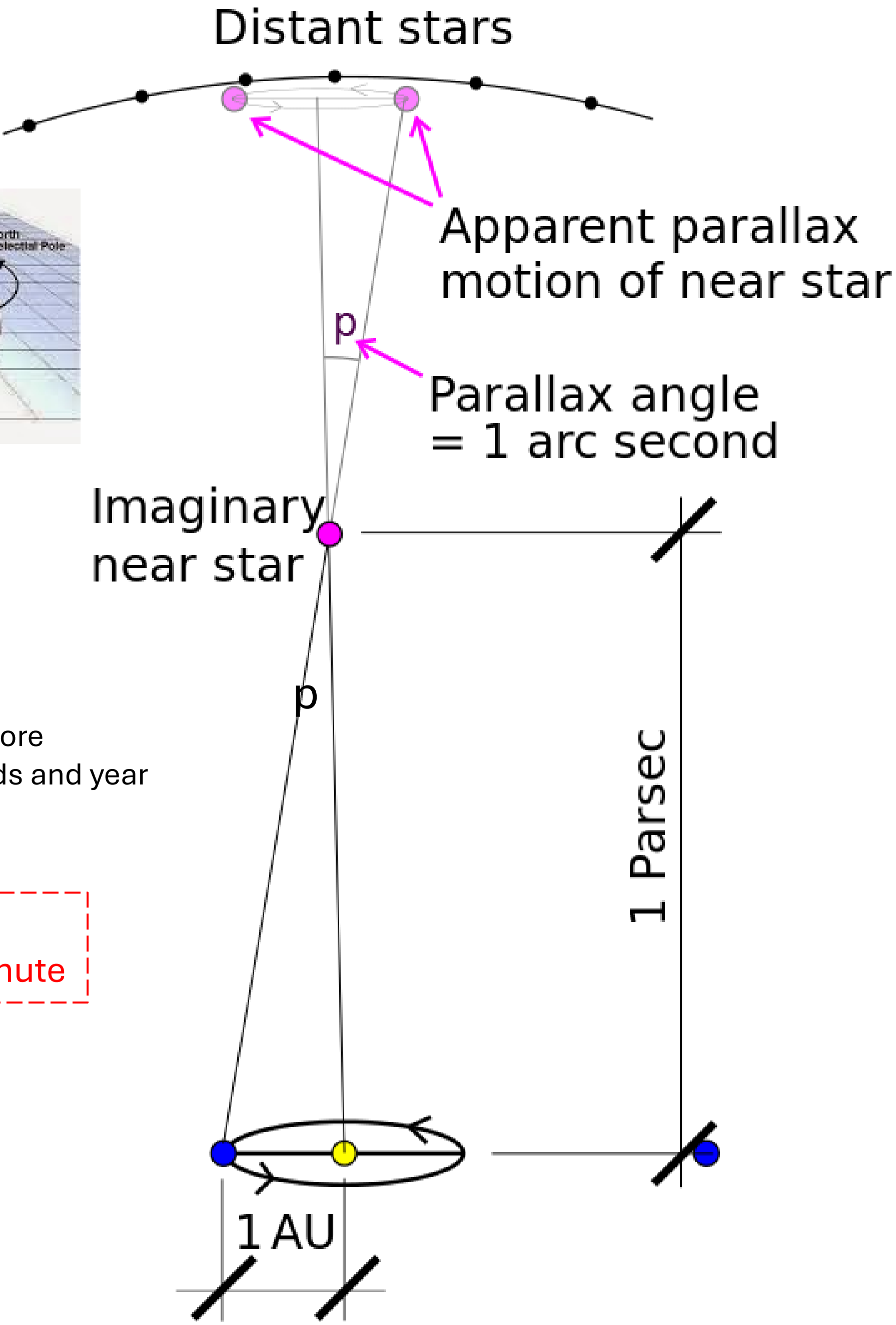
60 arc minutes = 1 degree
60 arc seconds = 1 arc minute

Parsec

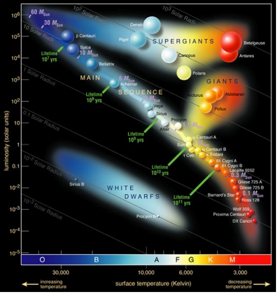
$$1\text{AU} = 1\text{pc}\times\tan\left(\frac{1^{\text{o}}}{60\times60}\right)$$

$$1\text{pc} = 2.063\times10^{5}\,\text{AU}$$

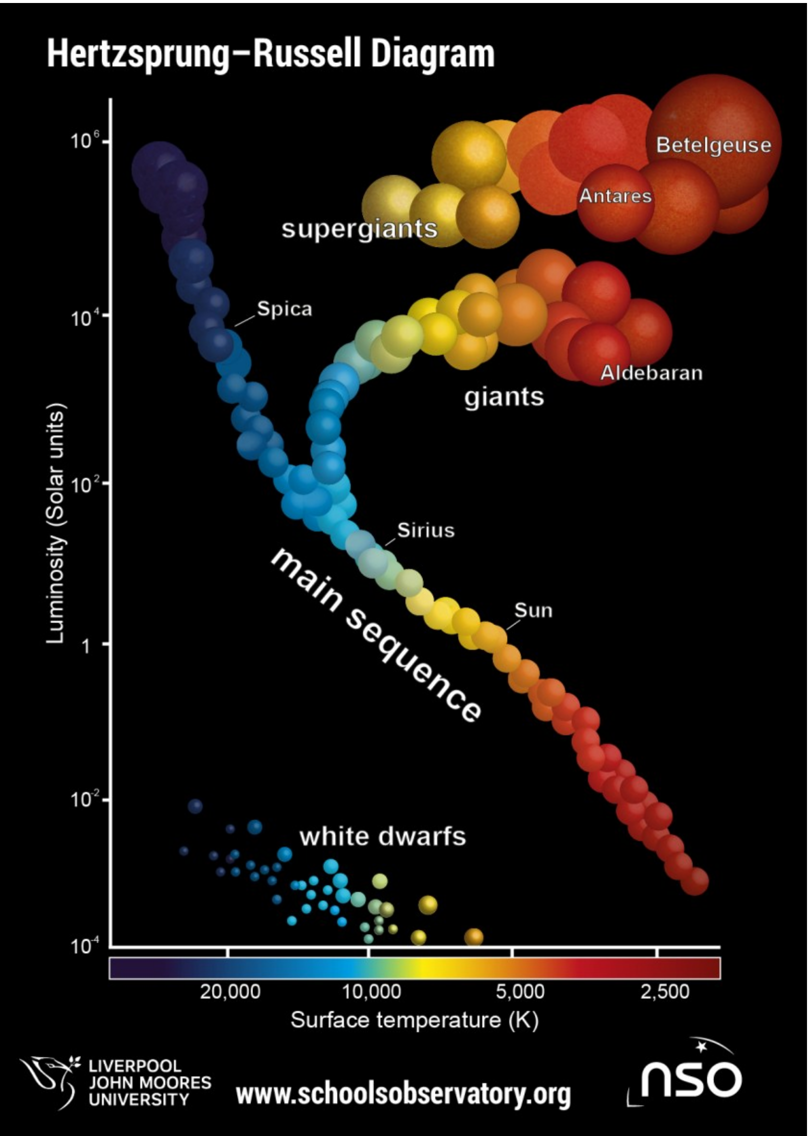
$$\boxed{1\text{pc} = 3.086\times10^{16}\,\text{m}}$$



Earth's motion around Sun

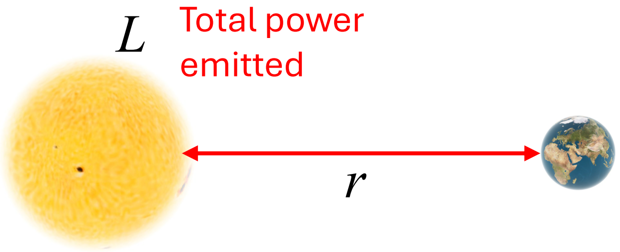


7. Calculating the distance to stars from Luminosity L and Colour $\lambda_{\max}$



Power per m² received from star is:

$$\boxed{\Phi = \frac{L}{4\pi r^2}}$$

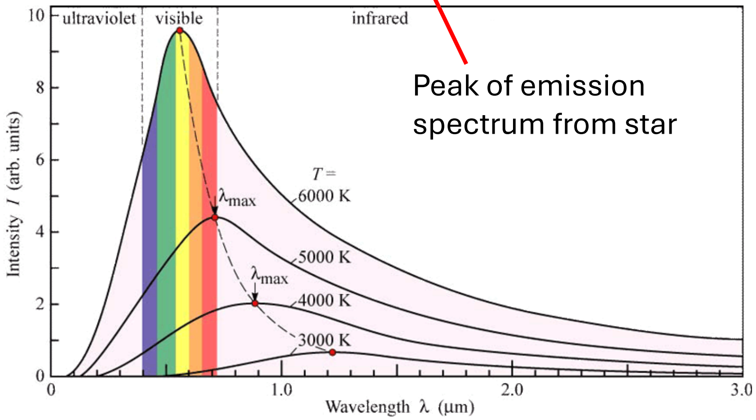


The HR diagram predicts **Luminosity** L of a star vs the **colour** of the star, which is related to its **surface temperature.**

$$\boxed{T \approx \frac{2{,}900\,\mu\mathrm{m}}{\lambda_{\max}}}$$

Wien's law

Wilhelm Wien
1864-1928



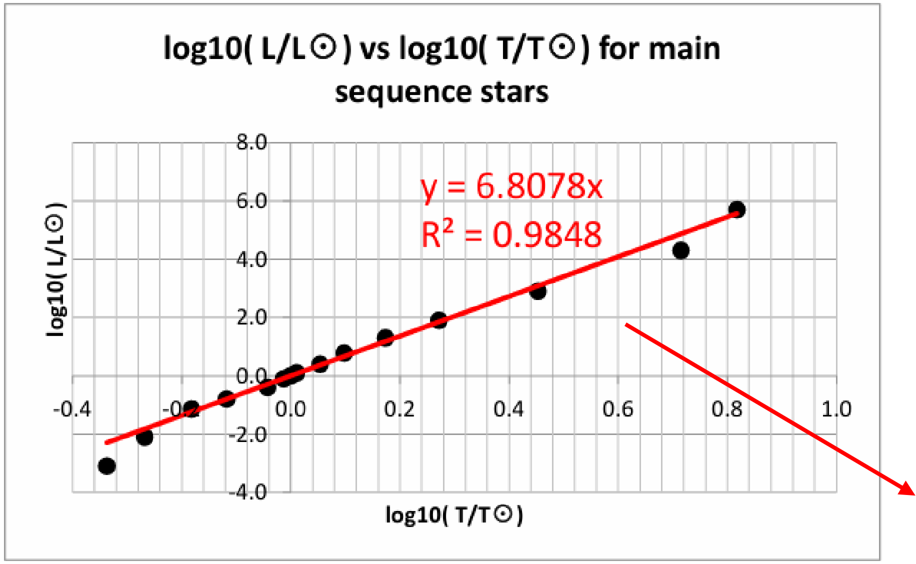
Peak of emission spectrum from star

$$\boxed{\therefore r = \sqrt{\frac{L}{4\pi\Phi}}}$$

Distance to star

MAIN SEQUENCE EXAMPLE STARS https://en.wikipedia.org/wiki/Main_sequence

Stellar classification	Radius R/R⊙	Mass M/M⊙	Luminosity L/L⊙	Surface temperature /K	Star	log10(L/L⊙)	log10 (T/T⊙)	log10(M/M⊙)	log10 ((R/R⊙)^2 * (T/5780K)^4)
M8	0.13	0.1	8.00E-04	2,660	Van Biesbroeck's star	-3.097	-0.337	-1.000	-3.120
M5	0.32	0.21	7.90E-03	3,120	EZ Aquarii A	-2.102	-0.268	-0.678	-2.061
M0	0.51	0.6	7.20E-02	3,800	Lacaille 8760	-1.143	-0.182	-0.222	-1.313
K5	0.74	0.69	1.60E-01	4,410	61 Cygni A	-0.796	-0.117	-0.161	-0.731
K0	0.85	0.78	4.00E-01	5,240	70 Ophiuchi A	-0.398	-0.043	-0.108	-0.312
G5	0.93	0.93	7.90E-01	5,610	Alpha Mensae	-0.102	-0.013	-0.032	-0.115
G2	**1**	**1**	**1.00E+00**	5,780	Sun	0.000	0.000	0.000	0.000
G0	1.05	1.1	1.26E+00	5,920	Beta Comae Berenices	0.100	0.010	0.041	0.084
F5	1.2	1.3	2.50E+00	6,540	Eta Arietis	0.398	0.054	0.114	0.373
F0	1.3	1.7	6.00E+00	7,240	Gamma Virginis	0.778	0.098	0.230	0.619
A5	1.7	2.1	2.00E+01	8,620	Beta Pictoris	1.301	0.174	0.322	1.155
A0	2.5	3.2	8.00E+01	10,800	Alpha Coronae Borealis A	1.903	0.271	0.505	1.882
B5	3.8	6.5	8.00E+02	16,400	Pi Andromedae A	2.903	0.453	0.813	2.971
B0	7.4	18	2.00E+04	30,000	Phi1 Orionis	4.301	0.715	1.255	4.599
O6	18	40	5.00E+05	38,000	Theta1 Orionis C	5.699	0.818	1.602	5.782



$$L_{\odot} = 4\pi R_{\odot}^{2}\sigma T^{4}$$

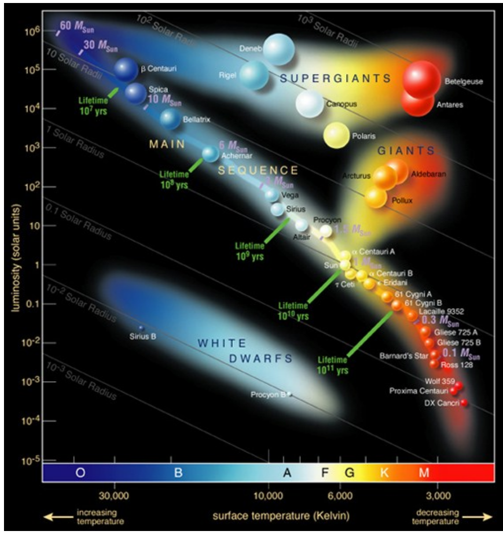
$$T = 5780\,\text{K}$$

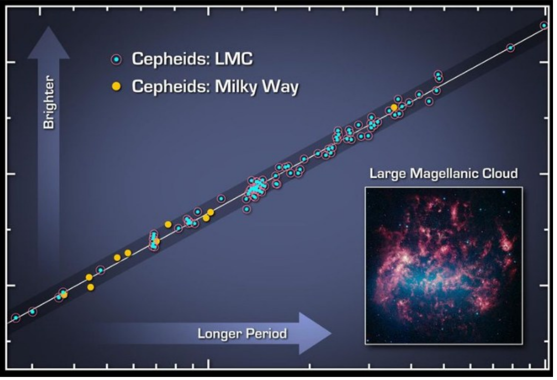
$$R_{\odot} = 6.96\times10^{8}\ \text{m}$$

$$\sigma = 5.67\times10^{-8}\ \text{Wm}^{-2}\text{K}^{-4}$$

$$\therefore \boxed{L_{\odot} \approx 3.83\times10^{26}\ \text{W}}$$

$$\boxed{L = L_{\odot}\times\left(\frac{T}{5790\,\text{K}}\right)^{6.81}}$$

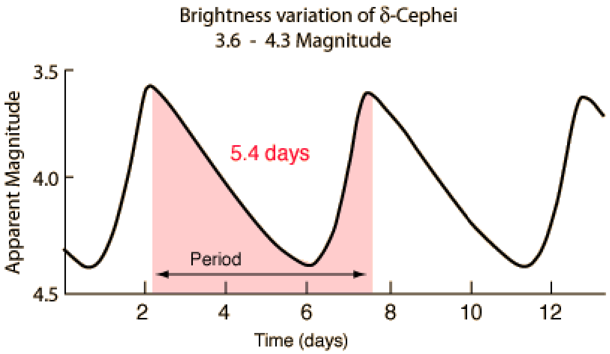


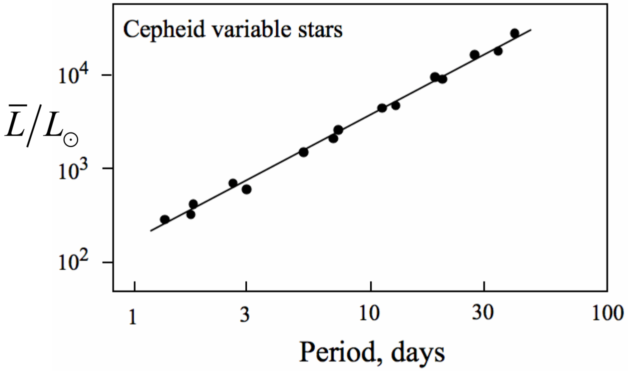


8. Calculating the distance to galaxies via Cepheid variable stars

H. Leavitt discovered (in 1908) that bright *variable stars* exhibit a correlation between average luminosity and period of variation. So measure star flux and period, and you can calculate the distance to the star.
This enables distance measurements (e.g. to nearby galaxies) up to 65 million light years.

RS Puppis, one of the brightest Cepheid variable stars in the Milky Way





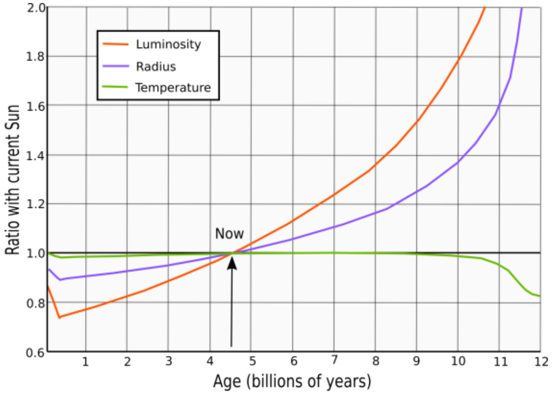
$$\log_{10}\left(\frac{\bar{L}}{L_\odot}\right) = 1.15\log_{10}\left(\frac{P}{\text{days}}\right) + 2.47$$

$$L_\odot = 3.828\times 10^{26}\ \text{W}$$

One Luminosity- period Cepheid correlation
There are several types!

The **pulsation mechanism** for Cepheids is thought to result from **double ionization of Helium**, which changes the opacity of a star, heating it up ... which causes it to expand, then cool ... ... which then reduces the Helium ionization, resulting in a **thermodynamic cycle.**

A. Eddington's model



Current solar Luminosity $L_\odot$ compared to earlier and predicted Epochs.

Henrietta Swan Leavitt
1898-1868

Star luminosity

Star radiant Flux in W/m².

$$\Phi = \frac{L}{4\pi d^2} \Rightarrow d = \sqrt{\frac{L}{4\pi\Phi}}$$

Distance to star

Luminosity to ***Bolometric magnitude*** M

$$L = 3.0128\times 10^{28}\ \text{W}\times 10^{-0.4M}$$

https://astronomy.swin.edu.au/sao/downloads/HET611-M17A01.pdf
https://en.wikipedia.org/wiki/Cepheid_variable

Cepheid variables – simply curious
Durham Astrolab

9. Calculating the scale of the observable Universe using Hubble’s law

H_0 is the ‘Hubble constant’, which has a modern value of about **71.9 km/s /Mpc**. It is *not really a constant*, as it relates to the *scale* of Universe expansion, which is thought not to be linear. The zero suffix therefore means ‘at the current epoch.’

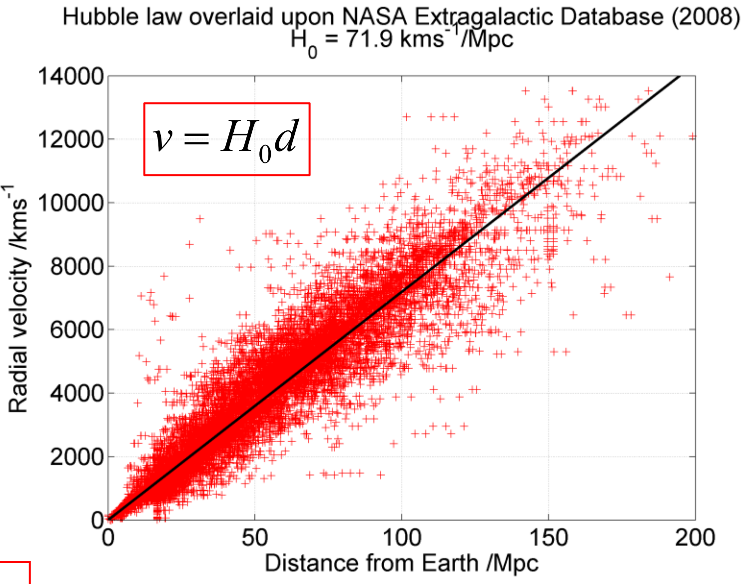
Edwin Hubble
1889-1953

Hubble’s law implies that *the Universe is expanding*. If we consider just the radial motion due to expansion (imagine a sponge being continuously enlarged, and tracking the relative distances between pairs of holes) and assume this is at a *constant* rate throughout time t , we can therefore make an **estimate of the age of the Universe.**

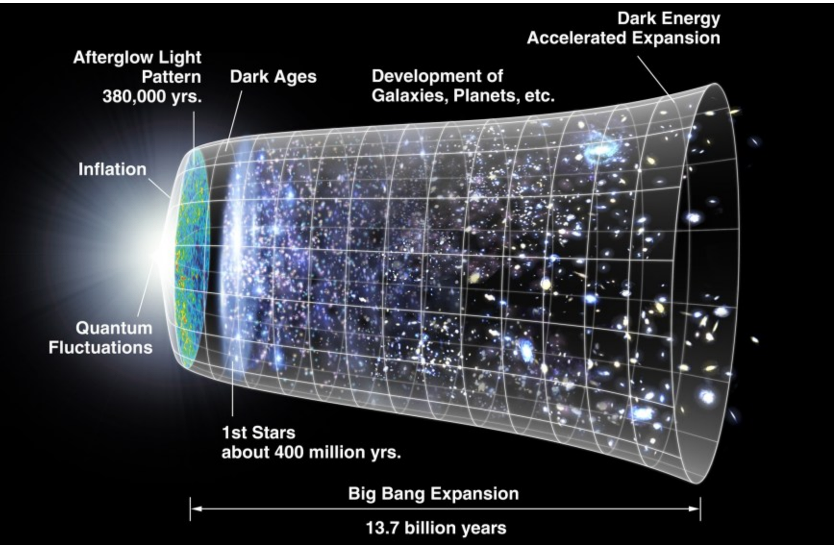
$$v = \frac{d}{t}, \qquad v = H_0 d \qquad \therefore H_0 d = \frac{d}{t} \quad \Rightarrow t = \frac{1}{H_0}$$

$$\therefore t = \left(\frac{71.9 \times 10^3 \,\text{ms}^{-1}}{3.086 \times 10^{22}\,\text{m}} \right)^{-1} = \boxed{13.6\,\text{billion years}}$$

1Mega-parsec (Mpc)
= 3.086 x10^{22} m



Edwin Hubble was perhaps the first astronomer to show that most galaxies (i.e. objects with distances of 10Mpc or more) have a recessional velocity v which is proportional to the distance d away from Earth*.



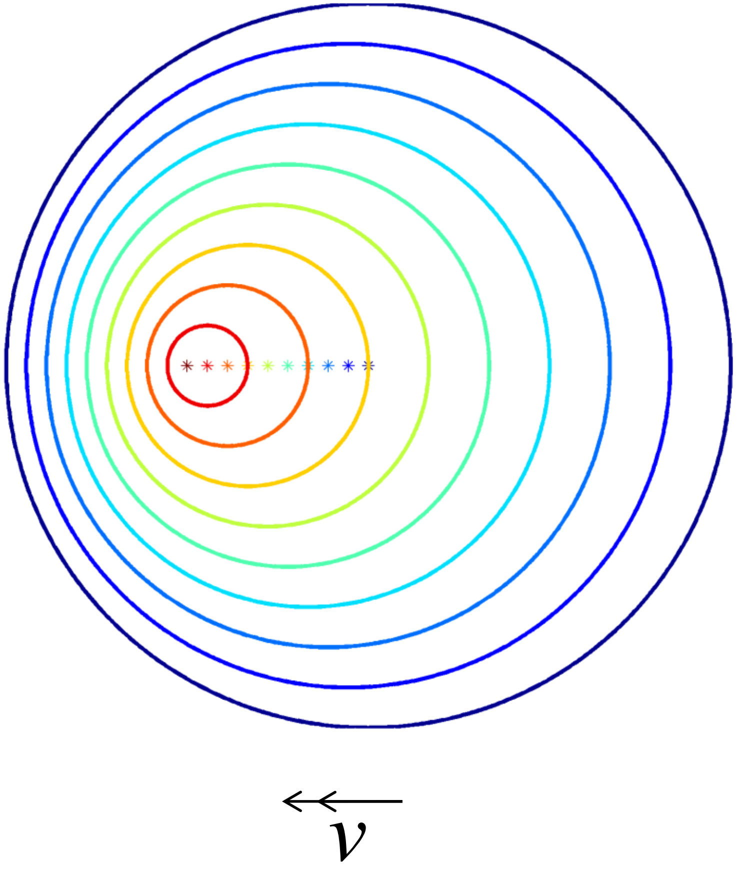
As of 2017, the best estimate for the age of the Universe is **13.799 +/- 0.021 billion years** using the **Lambda-CDM model** and observations of the **Cosmic Microwave Background** (CMB) radiation via **Planck** and **Wilkinson Microwave Anisotropy (WMAP)** probe (and others).

*The **Cosmological Principle** means *all parts of the Universe are expanding uniformly* relative to everywhere else, *at a given time* since the Big Bang. The Hubble law would therefore be the *same* from the perspective of a planet in another galaxy as it is on Earth.

Doppler shift method for measuring radial velocity

$$\boxed{c = f\lambda}$$

If an object emitting radiation at frequency f moves radially towards an observer at velocity v, the observer will measure a *slightly higher frequency* of radiation as the emitted waves ‘bunch up’.



Velocity of emitter *towards* observer

Frequency of emitted radiation

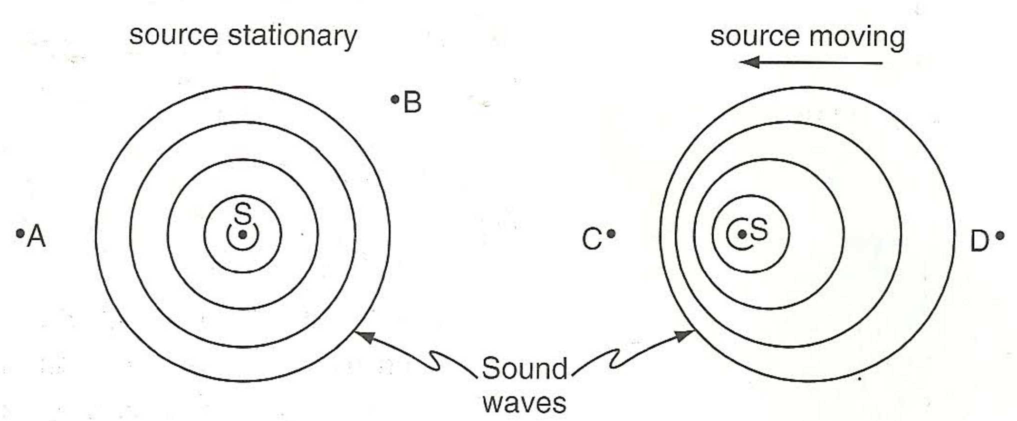
frequency change

$$\boxed{\Delta f = \frac{v}{c} f}$$

Speed of radiation

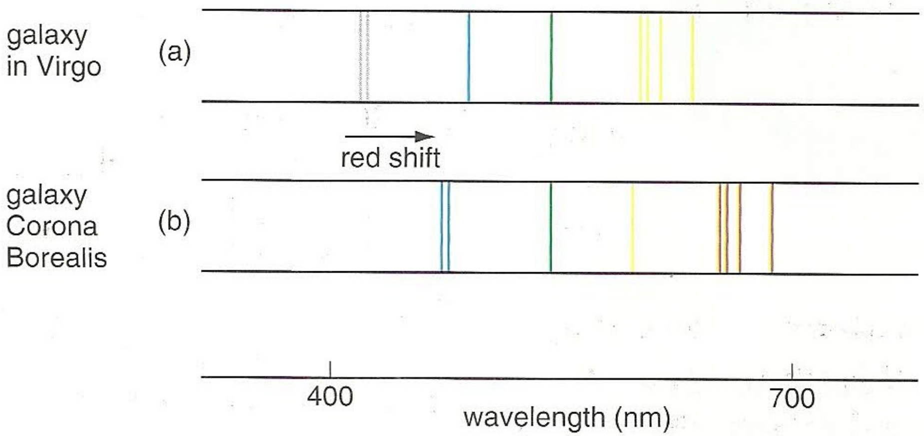
Note this formula is ‘Classical’. It is valid when $v \ll c$
Otherwise a **relativistic version** must be used.

Christian
Doppler
1803-1953

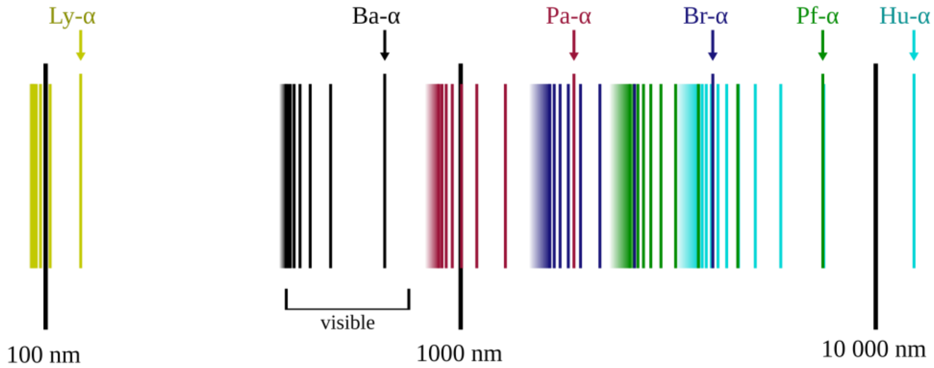


Redshift z is the fractional change in wavelength of light due to the doppler effect

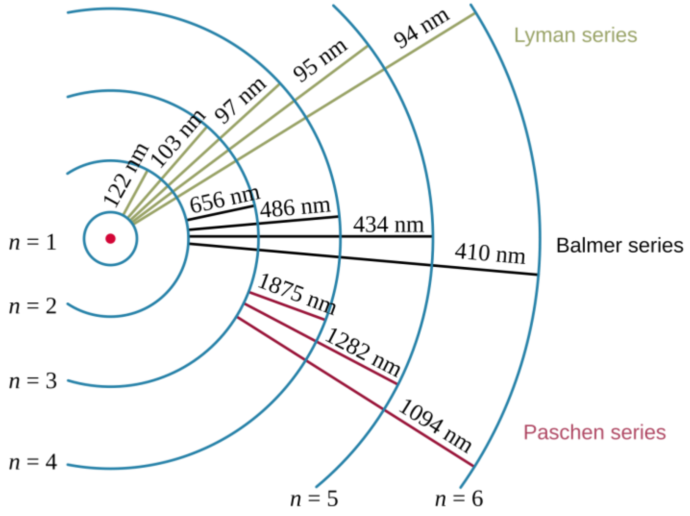
$$z = \frac{\lambda_{\text{observed}} - \lambda_{\text{emmitted}}}{\lambda_{\text{emmitted}}}$$



Compare spectral lines emissions from elements like Hydrogen and Helium from stars to those measured in the lab.

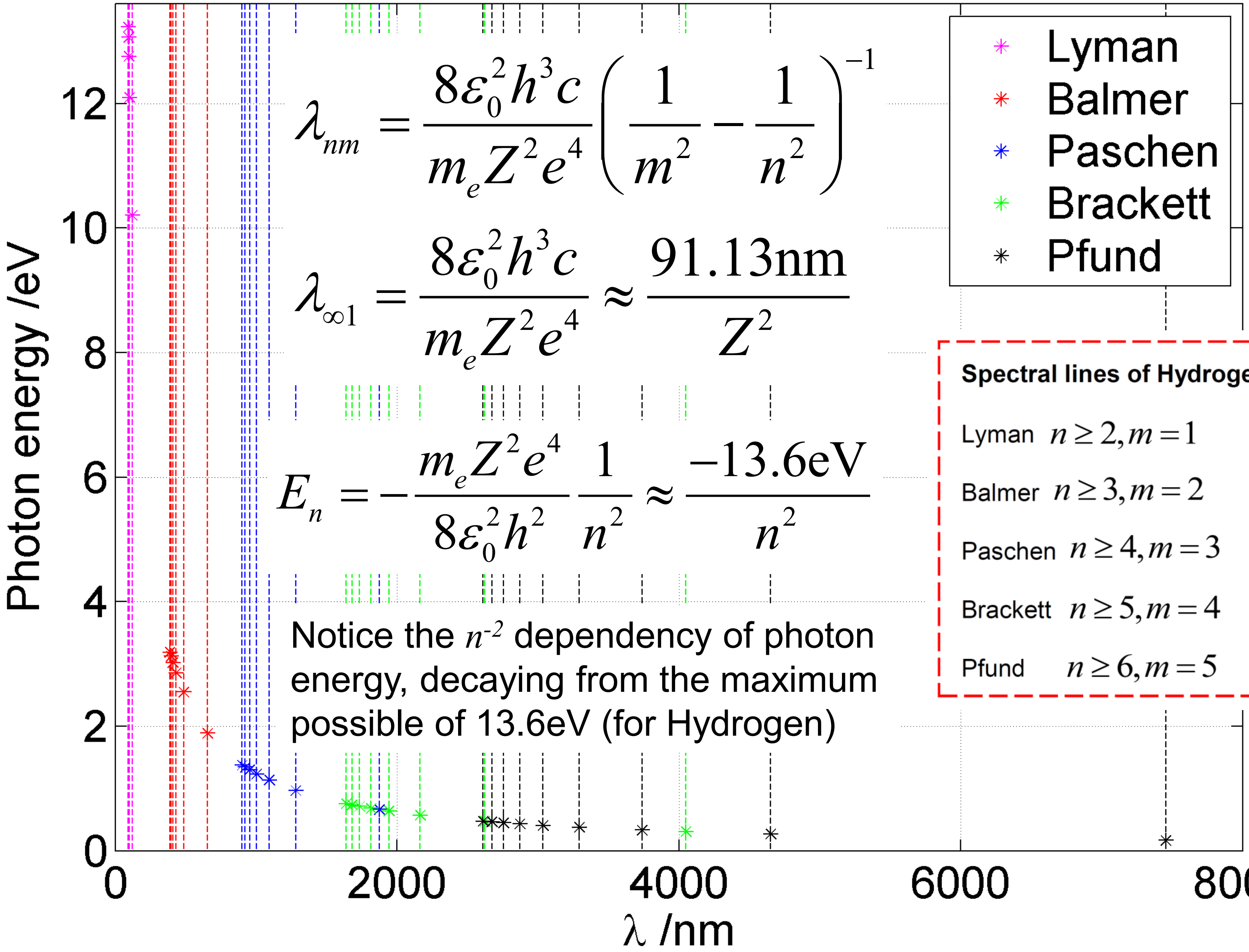


Hydrogen emission spectra



https://en.wikipedia.org/wiki/Hydrogen_spectral_series

Bohr model of Hydrogenic atom
photon emissions: Z = 1



$$\lambda_{nm} = \frac{8\varepsilon_0^2 h^3 c}{m_e Z^2 e^4}\left(\frac{1}{m^2}-\frac{1}{n^2}\right)^{-1}$$

$$\lambda_{\infty 1} = \frac{8\varepsilon_0^2 h^3 c}{m_e Z^2 e^4} \approx \frac{91.13\text{nm}}{Z^2}$$

$$E_n = -\frac{m_e Z^2 e^4}{8\varepsilon_0^2 h^2}\frac{1}{n^2} \approx \frac{-13.6\text{eV}}{n^2}$$

Notice the n^{-2} dependency of photon energy, decaying from the maximum possible of 13.6eV (for Hydrogen)

Spectral lines of Hydrogen $(Z=1)$

Lyman $n \geq 2, m = 1$

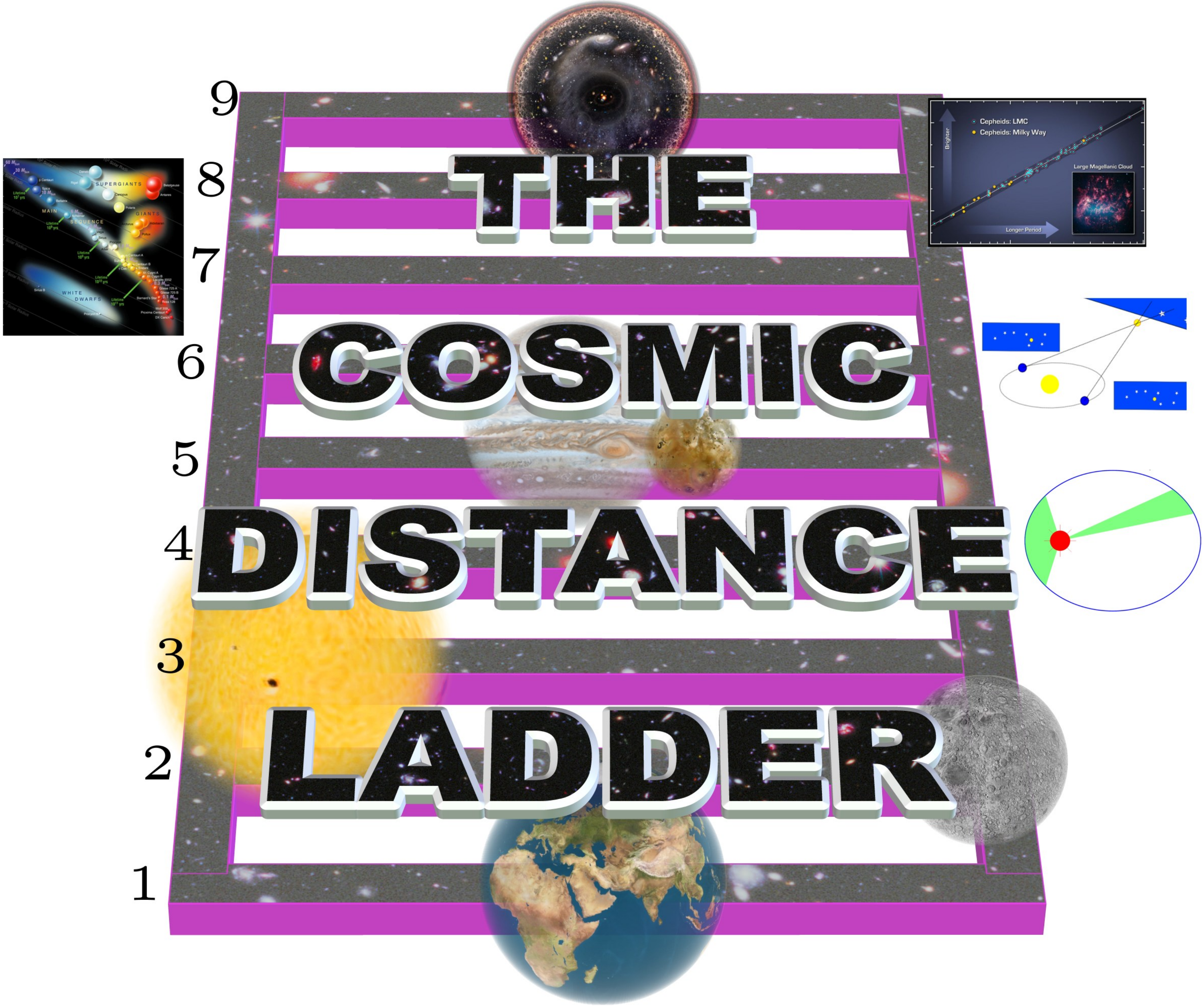
Balmer $n \geq 3, m = 2$

Paschen $n \geq 4, m = 3$

Brackett $n \geq 5, m = 4$

Pfund $n \geq 6, m = 5$

THE COSMIC DISTANCE LADDER



Dr Andrew French
2025