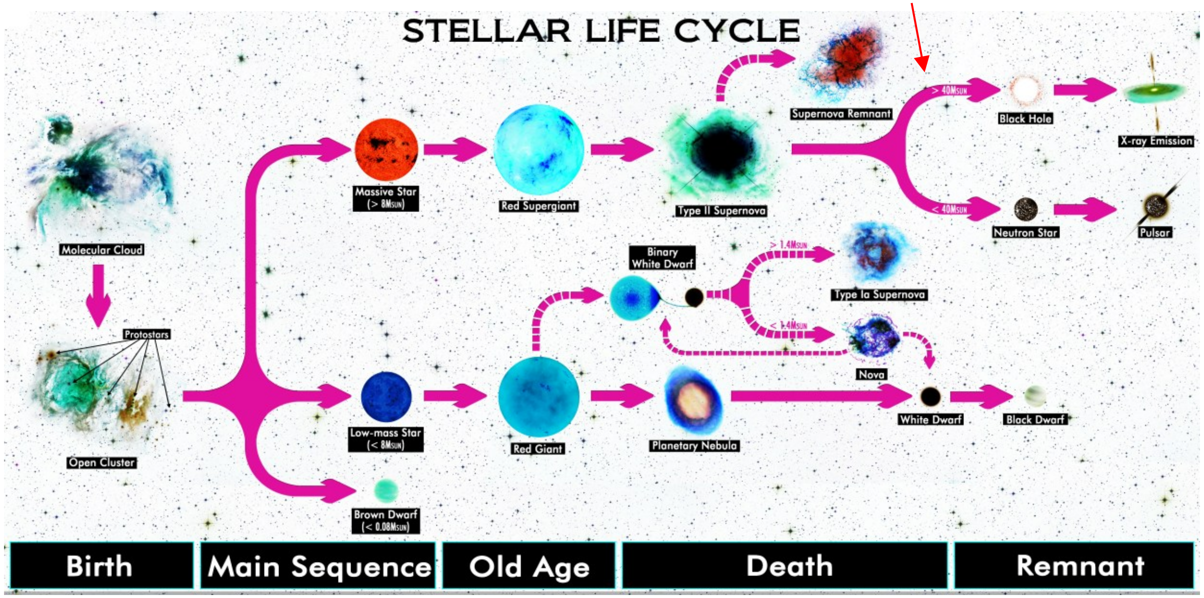
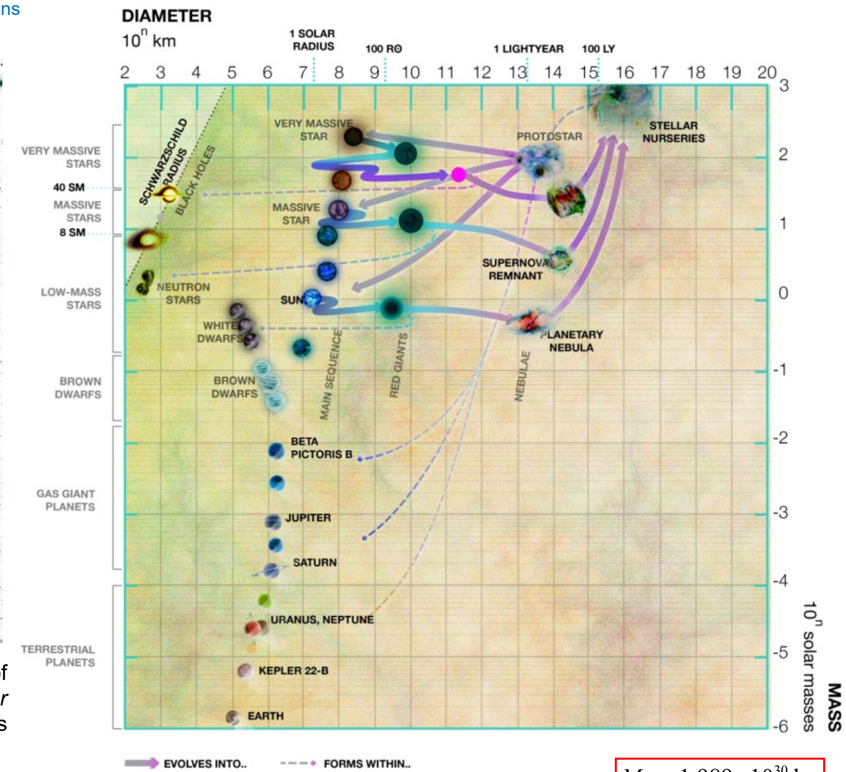
Cosmology: Structure and Evolution of Stars

This bit might not quite be correct! BH formation might be *direct* (i.e. sans supernova) for stars above about 40 solar masses.

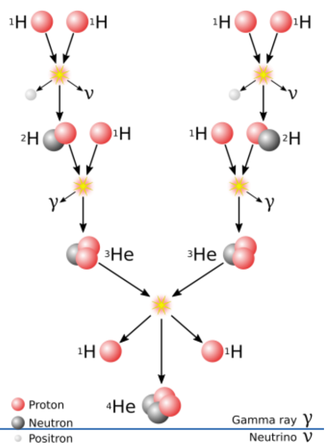




Protostars form from the gravitational collapse of gas and dust within **nebulae** or **molecular clouds** into rotating balls of (mostly) high temperature gas. If the protostar mass is less than about **0.08 solar masses**, its core temperature is *insufficient for nuclear fusion of hydrogen*. But if *deuterium fusion is theoretically possible*, these protostars will eventually be classified as **Brown Dwarfs**. These will shine dimly (mostly in the IR spectrum), and fade away slowly over hundreds of millions of years.

If the protostar mass is less than about **8 solar masses** (but more than 0.08), the core temperature will eventually reach about 10 million K, which is sufficient for the **p-p chain reaction** to initiate **hydrogen fusion.** This forms a **low-mass star** (like our Sun), which, after billions of years of stability, will eventually swell to a **red giant** and eventually dissipate, leaving a **white dwarf** (not really a star any more since fusion ceases) and ultimately a **black dwarf**. *Unless* the red giant forms a **binary** with a white dwarf and transfers mass such that the **white dwarf** exceed 1.44 solar masses. This results in a **Type 1a supernova**, with no remnant.

Chandrasekhar limit



$$4\,{}^1\mathrm{H}^+ + 2\,\mathrm{e}^- \rightarrow {}^4\mathrm{He}^{2+} + 2\,\nu_e$$

releasing 26.73 MeV of energy, some of which is lost to the neutrinos.

If the **protostar** is **over 8 solar masses**, it forms a **massive star**, which will eventually swell to form a **red (or perhaps blue)** ***supergiant***. Nuclear fusion progresses from hydrogen to heavier elements as its internal temperature increases, until it reaches iron. Iron fusion produces no net energy output, so thermal pressure is insufficient to counter gravitational collapse. If the star is **less than about 40 solar masses** it will undergo a **Type II supernova***. Above this mass, it is theorized that the star will collapse directly to a **Black Hole** *without a supernova*. If the **star is less than about 20 solar masses**, the core of the supernova remnant will form a **neutron star**. If this is highly magnetized and rotating, this neutron star will emit intense beams of EM radiation as a **pulsar**. **Between 20 and 40 solar masses**, the star will undergo a **Type II supernova** and the remnant will collapse to form a **Black Hole**. A **supermassive black hole** (millions of billions of solar masses) will accrete rings of gas, and can produce intense jets of EM radiation. Most galactic centres are thought to be supermassive black holes, and electromagnetically active ones are called **quasars**.

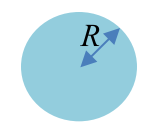
*The energy in this violent supernova process may be sufficient to synthesize heavier elements than iron.

Nuclear fusion within a star will result in **radiation pressure** from the emitted photons. For stars above about 120 solar masses, radiation pressure would be so extreme that gravitational stability is though to be unlikely. Although perhaps exceptions exist!

So **star** masses (excluding Black Holes) should be within the range of **0.08 and about 120 solar masses**.

$M_\odot = 1.989 \times 10^{30}$ kg
$R_\odot = 696{,}340$ km
$L_\odot = 3.846 \times 10^{26}$ Js^{-1}
$T_\odot = 5778$ K
$\mathrm{AU} = 1.496 \times 10^{11}$ m
$1\mathrm{ly} = 9.461 \times 10^{15}$ m

Radiation produced from nuclear fusion in the core of a star will be constantly reabsorbed, so it can take a long time for these photons to escape. The total about of radiative power of a star is called its **luminosity** L. If the **effective temperature** is T_e and the star radius is R :



$$L = 4\pi R^2 \sigma T_e^4$$

Not the same as surface temperature!

$$\sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2}$$

$k_B = 1.381 \times 10^{-23}\,\mathrm{m^2kgs^{-2}K^{-1}}$ Boltzmann's constant
$h = 6.626 \times 10^{-34}\,\mathrm{m^2kgs^{-1}}$ Planck's constant
$c = 2.998 \times 10^8\,\mathrm{ms^{-1}}$ Speed of light
$\sigma = 5.67 \times 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$ Stefan-Boltzmann constant

Stars in the Main Sequence

https://chandra.harvard.edu/edu/formal/variable_stars/bg_info.html

Once a star has formed, i.e. nuclear fusion commences in its core, a star will typically exist in a quasi-static state in what is known as the **Main Sequence (MS)** for a time τ which depends on the ratio of its mass M to its luminosity L.

$$\tau \approx 10^{10}\text{ yr}\times\left(\frac{M}{M_\odot}\right)\left(\frac{L}{L_\odot}\right)^{-1}$$

$$L \propto M^{3.5}$$

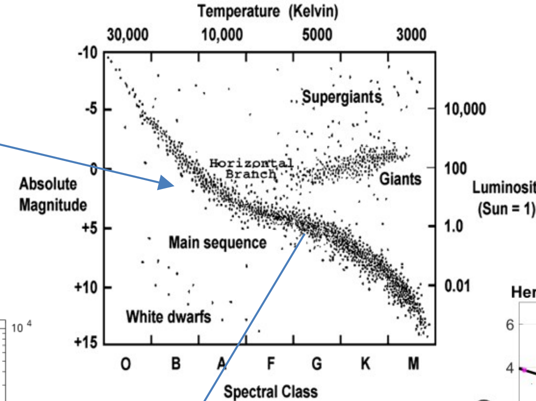
$$\therefore \tau \approx 10^{10}\text{ yr}\times\left(\frac{M}{M_\odot}\right)^{-2.5}$$

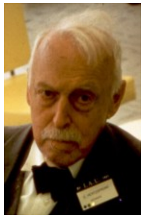
So more massive stars have much shorter lifetimes

https://en.wikipedia.org/wiki/Main_sequence#Evolutionary_tracks

If one plots a log, log graph of luminosity vs effective temperature, (this is called A **Hertzsprung Russell diagram**) most stars are clustered along a diagonal line. This is the MS. At later stages of stellar evolution, stars will 'meander' off the MS and branch off to giants, (or supergiants) and then possibly white dwarfs.(or black holes).

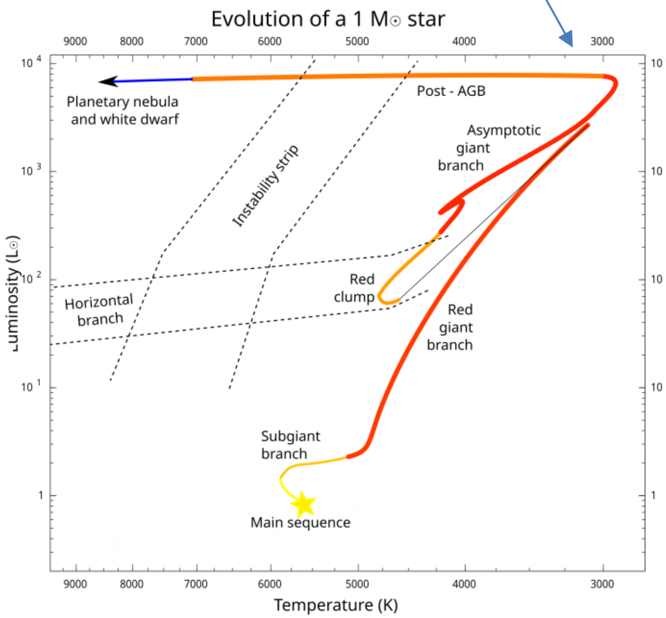
Hertzsprung-Russell (HR) diagram





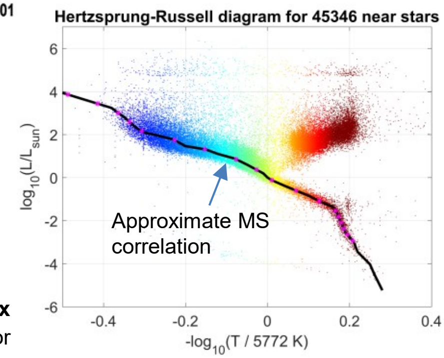
Ejnar Hertzsprung (1873-1967)

Henry Norris Russell (1877-1957)



The **MS correlation** means L can be predicted from T_e (assuming a star is in the MS), which means both the radius and the distance to the star d can be calculated. The latter can be found by measuring the **radiation flux** Φ (in W/m2) from a star by an Earth or near-Earth telescope.

Hertzsprung-Russell diagram for 45346 near stars



$$T_s = \frac{hc}{4.9651 k_B \lambda_{max}}$$

$$L = 4\pi R^2 \chi\sigma T_e^4 \Rightarrow R = \sqrt{\frac{L}{4\pi\chi\sigma T_e^4}}$$

Find star surface temperature from peak of Planck spectrum for star.

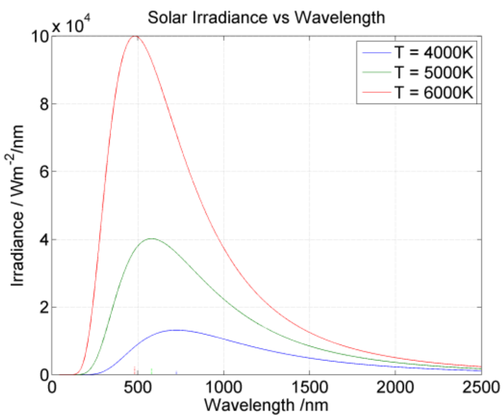
Calculate star radius from luminosity and effective temperature.

Note emissivity χ might not be 1 for all stars..

$$\Phi = \frac{L}{4\pi d^2} \Rightarrow d = \sqrt{\frac{L}{4\pi\Phi}}$$

Find distance to star if know the luminosity and measure the radiation flux.

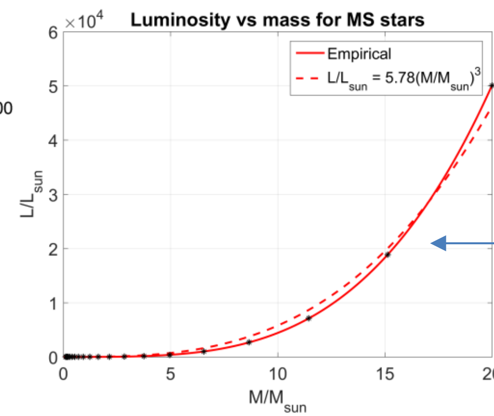
The **surface temperature** of a star is related to the wavelength of the peak of the spectrum of **solar irradiance** (i.e. the *Planck spectrum*). The latter (i.e. 'colour') can be measured for stars, and hence effective temperature T_e can be calculated.



$$I = \int_0^\infty B(\lambda,T)\,d\lambda = \sigma T^4,\quad \sigma = \frac{2\pi^5 k_B^4}{15c^2h^3}$$

$$B(\lambda,T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_B T}}-1}$$

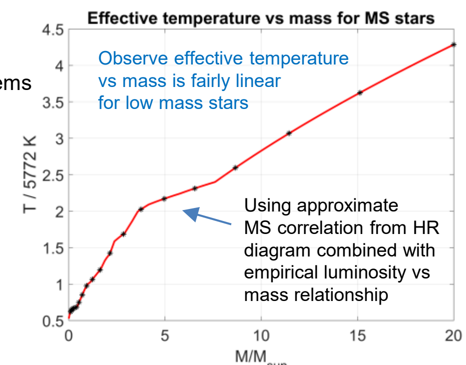
$$\lambda_{max} = \frac{hc}{4.9651 k_B T} \approx \frac{2.9\times10^6\text{ nm}}{(T/\text{K})}$$



Empirical **luminosity vs star mass relationship**
For MS stars, using measurements from binary star systems

$$\frac{L}{L_\odot} \approx \begin{cases} 0.23(M/M_\odot)^{2.3} & M/M_\odot < 0.43 \\ (M/M_\odot)^4 & 0.43 < M/M_\odot < 2 \\ 1.4(M/M_\odot)^{3.5} & 2 < M/M_\odot < 55 \\ 32{,}000\,M/M_\odot & M/M_\odot > 55 \end{cases}$$

https://en.wikipedia.org/wiki/Mass%E2%80%93luminosity_relation



$\sigma = 5.67\times10^{-8}\text{ Wm}^{-2}\text{K}^{-4}$ $\quad c = 2.99\times10^{8}\text{ ms}^{-1}$ $\quad k_B = 1.38\times10^{-23}\text{ JK}^{-1}$

Physical properties of the interior of (Main Sequence) stars: Polytropic, ideal gas model

We can describe a quasi-static, spherically symmetric star using the following set of differential equations.

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

Mass build up via spherical shells

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$

Hydrostatic equilibrium between **gravity** and gas (and radiation) **pressure**

$$\frac{dl}{dr} = 4\pi r^2 \rho \varepsilon$$

Power per unit area of internal shell inside star generated via **nuclear fusion**

Boundary conditions:

$$M = \int_0^R dm = \int_0^R 4\pi r^2 \rho dr$$

$$L = 4\pi R^2 \sigma T_e^4$$

Assume emissivity χ is unity

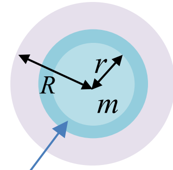
Note T_e is *not* the same as surface temperature!

$0 \le r \le R$

$$\frac{dm}{dr} = 4\pi r^2 \rho$$ Sum spherical shells

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$ Newton's law of Gravitation.

$$dP = -\frac{dF}{4\pi r^2} \therefore dF = \frac{Gm}{r^2} \times 4\pi r^2 dr \rho$$



Gravitational force between mass or radius r and shell of width dr above it. For equilibrium, this must be balanced by gas (and radiation) pressure x shell area.

ε is the **energy generated per unit mass** via nuclear fusion (excluding neutrino production, which is assumed to leave the star and not interact with higher radius layers).

For low-mass, low temperature stars the **pp-chain** is the primary mode of nuclear fusion

$$\varepsilon = \eta_{pp} X^2 \rho T^4$$

For higher temperatures, the **CNO cycle** is possible

$$\varepsilon = \eta_{CNO} X X_{CNO} \rho T^{17}$$

And for even higher temperatures the **triple alpha** fusion reaction can occur

$$\varepsilon = \eta_{\alpha\alpha\alpha} Y^3 \rho^2 T^{40}$$

Note very strong T dependence!

$${}^4_2\text{He} + {}^4_2\text{He} \rightarrow {}^8_4\text{Be} + \gamma$$

$${}^4_2\text{He} + {}^8_4\text{Be} \rightarrow {}^{12}_6\text{C} + \gamma$$

Beyond this, **fusion involving carbon, oxygen and eventually silicon is possible**. But when iron is produced, fusion is no longer possible in a star. To synthesize heavier elements you need a supernova! Note as T increases the fraction of net fusion energy carried away by neutrinos increases.

Fuel	Process	T_{thresh}[1] (K)	Products	E/nucleon[2] (MeV)	Timescale[3] (yr)
H	p-p	$\sim 4 \times 10^6$	He	6.55	
H	CNO	1.5×10^7	He	6.25	1×10^7
He	triple-α	1×10^8	C, O	0.61	1×10^6
C	C + C	6×10^8	O, Ne, Na, Mg	0.54	300
O	O + O	1×10^9	Mg, S, P, Si	~ 0.3	0.5
Si	Nucl. equil.	3×10^9	Co, Fe, Ni	$\lesssim 0.2$	0.005 (2 days!)

Let us model a star as a **polytropic gas** with pressure vs density variation

$$P = P_0 \left(\frac{\rho}{\rho_0}\right)^{1+\frac{1}{n}}$$

Rocky planets have a fixed density so $n = 0$.
Neutron stars are modelled with n between 0.5 and 1.
Red giants, brown dwarfs, gas giant planets (like Jupiter) and *also* **low-mass white dwarfs** have $n = 1.5$.
Higher mass white dwarfs and **MS stars** have $n = 3$.
$n = 5$ implies an **infinite radius.**
n = infinite implies an *isothermal sphere* (e.g. models a **globular cluster of galaxies**).

https://en.wikipedia.org/wiki/Polytrope

Inputs are just M, X, Y

L, T_s from MS empirical formulae

$$R = \sqrt{\frac{L}{4\pi\sigma T_s^4}}$$

For the **simplest model** of a MS star we will assume the same polytropic relationship thought the star, so P_0 and ρ_0 correspond to the **core** of the star at $r = 0$. More sophisticated models may involve distinct radial regions, such as a helium core surrounded by hydrogen 'burning' shells.

We will also assume the star is an **ideal gas**, and we can **ignore radiation pressure**.

$$P = \frac{\rho_0 k_B T_0}{\mu}\left(\frac{\rho}{\rho_0}\right)^{1+\frac{1}{n}}$$

$$T = \frac{P\mu}{k_B \rho}$$

The **Lane-Emden model** can be used to determine the mass, pressure, density and temperature variations with radius. (See next page!)

Pressure can be modelled by **the ideal gas equation** (force per unit area results from random collisions between molecules) and radiation pressure (compressive effect Resulting from impinging radiation, its reflection plus 'black body' radiation of the gas itself)

$$P = \frac{\rho k_B T}{\mu} + \frac{4\sigma}{3c} T^4$$

Ideal gas

Radiation pressure

$k_B = 1.38 \times 10^{-23}\ \text{JK}^{-1}$

Average molar mass of star matter depends on **hydrogen (mass) fraction (X)**, and **Helium fraction (Y).** The remainder (**Z**) is known as '**metallicity**'.

We often assume star interiors are *fully ionized*.

$$X + Y + Z = 1$$

$$\mu = m_p \left(2X + \tfrac{3}{4}Y + \tfrac{1}{2}Z\right)^{-1} \quad **$$

$$\therefore \mu = m_p \left(2X + \tfrac{3}{4}Y + \tfrac{1}{2}(1 - X + Y)\right)^{-1}$$

$$\therefore \mu = m_p \left(\tfrac{3}{2}X + \tfrac{1}{4}Y + \tfrac{1}{2}\right)^{-1}$$

$$\therefore \mu = \frac{4m_p}{6X + Y + 2} \rightarrow 0.5 \le \mu \le 2$$

For the Sun $X = 0.747, Y = 0.236, Z = 0.017$

$$\therefore \mu \approx 0.6 m_p$$

** The idea is to count the ionized particles per nucleon. Hydrogen produces two (e^- + p^+), Helium produces three ($2e^-$ + He^{2+}) of four and one assumes most 'metals' are approximately equal numbers of protons and neutrons.

So assuming all particles have the same KE (and hence contribute equally to the ideal gas pressure), the mass of this particle is the proton mass / number of ionized particles per proton (or neutron).

https://vikdhillon.staff.shef.ac.uk/teaching/phy213/phy213_molecular.html

Molar mass of a proton $m_p = 6.0221 \times 10^{23} \times 1.6726 \times 10^{-27}\ \text{kgmol}^{-1} \approx 1.007\ \text{kgmol}^{-1}$

Lane-Emden model of a polytropic star

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}, \quad P = a\rho^{1+\frac{1}{n}}, \quad 0 \le r \le R$$

Hydrostatic equilibrium between **gravity** and gas (and radiation) **pressure**

$$\therefore m = -\frac{r^2}{G\rho}\frac{dP}{dr} = -\frac{r^2 a}{G\rho}\left(\frac{n+1}{n}\right)\rho^{\frac{1}{n}}\frac{d\rho}{dr}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

Mass build up via spherical shells

$$a = \frac{k_B T_0}{\mu}\rho_0^{-\frac{1}{n}}$$

if assume Ideal gas

$$\therefore -\frac{a}{G}\left(\frac{n+1}{n}\right)\frac{d}{dr}\left(r^2\rho^{\frac{1}{n}-1}\frac{d\rho}{dr}\right) = 4\pi r^2 \rho$$

$$\therefore \frac{1}{r^2\rho}\frac{d}{dr}\left(r^2\rho^{\frac{1}{n}-1}\frac{d\rho}{dr}\right) = -\frac{4\pi G}{a}\left(\frac{n}{n+1}\right)$$

Define another set of variables $\rho = \rho_0\theta^n, \; r = \alpha\psi$

Where the **core density** at $r = 0$ (and $\theta = 1$) is ρ_0

$$\therefore \frac{1}{\alpha^2\psi^2\rho_0\theta^n}\frac{d}{d\psi}\left(\psi^2\rho_0^{\frac{1}{n}-1}\theta^{1-n}\rho_0\frac{d\theta^n}{d\psi}\right) = -\frac{4\pi G}{a}\left(\frac{n}{n+1}\right)$$

$$\therefore \frac{1}{\psi^2}\frac{d}{d\psi}\left(\psi^2\rho_0^{\frac{1}{n}}\theta^{1-n}n\theta^{n-1}\frac{d\theta}{d\psi}\right) = -\frac{4\pi G}{a}\left(\frac{n}{n+1}\right)\alpha^2\rho_0\theta^n$$

$$\therefore \frac{1}{\psi^2}\frac{d}{d\psi}\left(\psi^2\frac{d\theta}{d\psi}\right) = -\frac{4\pi G}{a}\left(\frac{1}{n+1}\right)\alpha^2\rho_0^{1-\frac{1}{n}}\theta^n$$

At the centre of the star

$r = 0, \; \rho = \rho_0, \; \theta = 1$

$r = \alpha\psi \Rightarrow \psi = 0$

At the surface of the star

$r = R, \; \rho = 0, \; \psi = \psi_0$

$\rho = \rho_0\theta^n \Rightarrow \theta = 0$

Let's define constant α such that: $1 = \frac{4\pi G}{a}\left(\frac{1}{n+1}\right)\alpha^2\rho_0^{1-\frac{1}{n}}$

$$\therefore \alpha = \sqrt{\frac{a(n+1)}{4\pi G}\rho_0^{\frac{1}{n}-1}} = \sqrt{\frac{k_B T_0}{4\pi G\mu}\rho_0^{-\frac{1}{n}}(n+1)\rho_0^{\frac{1}{n}-1}}$$

$$\therefore \alpha = \sqrt{\frac{k_B T_0(n+1)}{4\pi G\mu\rho_0}}$$

$$\therefore T_0 = \frac{4\pi G\mu\rho_0\alpha^2}{k_B(n+1)} = \frac{4\pi G\mu\rho_0 R^2}{k_B(n+1)\psi_0^2}$$

Hence: $$\frac{1}{\psi^2}\frac{d}{d\psi}\left(\psi^2\frac{d\theta}{d\psi}\right) = -\theta^n$$

This is called the **Lane-Emden equation**

https://en.wikipedia.org/wiki/Lane%E2%80%93Emden_equation

J.H Lane, R. Emden

Neutron stars are modelled with n between 0.5 and 1.
Red giants, brown dwarfs, gas giant planets (like Jupiter) and *also* **low-mass white dwarfs** have $n = 1.5$.
Higher mass white dwarfs and **MS stars** have $n = 3$.
$n = 5$ implies an infinite radius.
n = infinite implies an *isothermal sphere*

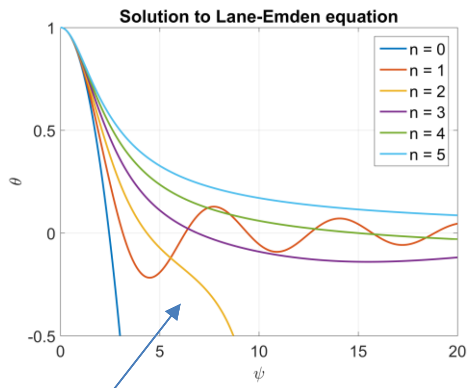
Some analytic solutions exist ($n = 0$, $n = 1$, $n = 2$ and in certain regions, $n = 5$),
But it can also be solved via a numeric method.

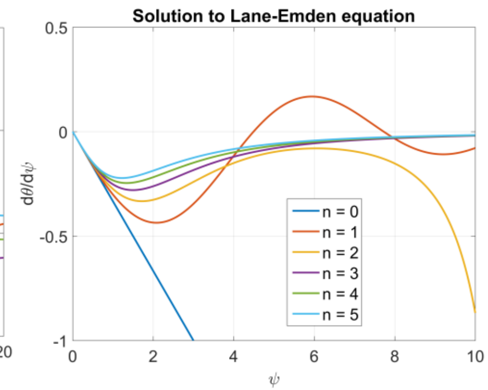
Define $$\xi = -\psi^2\frac{d\theta}{d\psi} \Rightarrow \frac{d\theta}{d\psi} = -\frac{\xi}{\psi^2}$$

Therefore the Lane-Emden equation becomes $$\frac{d\xi}{d\psi} = \psi^2\theta^n$$

A first-order 'Euler' solution method is:

$$\psi_0 = 0, \; \xi_0 = 0, \; \theta_0 = 1, \; \Delta\psi = 0.001$$
$$\psi_{i+1} = \psi_i + \Delta\psi$$
$$\Delta\xi = \psi_i^2\theta_i^n\Delta\psi$$
$$\xi_{i+1} = \xi_i + \Delta\xi$$
$$\Delta\theta = -\frac{\xi_i}{\psi_i^2}\Delta\psi$$
$$\theta_{i+1} = \theta_i + \Delta\theta$$





Terminate iteration when $\theta_{i+1} < 0$

... Although for $n = 4$ or $n = 5$ this may result in an infinite loop. A practical method is to set a sensible upper limit such as $\psi_{i+1} \le 10$

Some special cases:

$n = 0$:	$\theta = 1 - \frac{1}{6}\psi^2$	$\frac{d\theta}{d\psi} = -\frac{1}{3}\psi$	$\psi_0 = \sqrt{6}$
$n = 1$:	$\theta = \frac{\sin\psi}{\psi}$	$\frac{d\theta}{d\psi} = \frac{\psi\cos\psi - \sin\psi}{\psi^2}$	$\psi_0 = \pi$

Calculating mass vs radius

$$\frac{dm}{dr} = 4\pi r^2\rho, \quad \rho = \rho_0\theta^n, \; r = \alpha\psi$$

$$\therefore \frac{1}{\alpha}\frac{dm}{d\psi} = 4\pi\alpha^2\psi^2\rho_0\theta^n$$

$$\therefore m = 4\pi\alpha^3\rho_0\int_0^\psi \phi^2\theta^n d\phi$$

$$\psi^2\theta^n = -\frac{d}{d\psi}\left(\psi^2\frac{d\theta}{d\psi}\right)$$

$$\therefore m = -4\pi\alpha^3\rho_0\int_0^\psi \frac{d}{d\phi}\left(\phi^2\frac{d\theta}{d\phi}\right)d\phi$$

$$\therefore m = 4\pi\alpha^3\rho_0\left[-\psi^2\frac{d\theta}{d\psi}\right]$$

If we restrict n to 3 or less (i.e. modelling most star types) then have a finite radius and mass.

$r = R, \; m = M, \; \theta = 0, \; \psi = \psi_0$

$$\Lambda = -\psi_0^2\left.\frac{d\theta}{d\psi}\right|_{\psi_0} \quad M = 4\pi\alpha^3\rho_0\Lambda \quad R = \alpha\psi_0$$

n	0	1	2	3	4	5
ψ_0	$\sqrt{6}$	π	4.35	6.89	14.97	∞
$-\psi_0^2\left.\frac{d\theta}{d\psi}\right\|_{\psi_0}$	$2\sqrt{6}$	π	2.41	2.02	1.80	1.7
$\frac{\rho_0}{\bar{\rho}}$	1	$\frac{1}{3}\pi^2$	11.40	54.16	621.92	∞

The average density is $\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$

$$\therefore \bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{4\pi\alpha^3\rho_0\Lambda}{\frac{4}{3}\pi\alpha^3\psi_0^3} = \frac{3\rho_0\Lambda}{\psi_0^3}$$

$$\therefore \frac{\rho_0}{\bar{\rho}} = \frac{\psi_0^3}{3\Lambda}$$

This is called the '*condensation*'

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} \Rightarrow \rho_0 = \frac{\psi_0^3}{3\Lambda}\bar{\rho} \approx 54.16\frac{M}{\frac{4}{3}\pi R^3}$$

$$R = \alpha\psi_0 \Rightarrow T_0 = \frac{4\pi G\mu\rho_0 R^2}{k_B(n+1)\psi_0^2}$$

Compute core density and temperature from M, R

$G = 6.674 \times 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2}$ Universal gravitational constant

Mass vs radius relationship for polytropes

$$P = a\rho^{1+\frac{1}{n}}$$

$$R = \alpha\psi_0$$
$$M = 4\pi\alpha^3\rho_0\Lambda$$
$$\Lambda = -\psi_0^2 \left.\frac{d\theta}{d\psi}\right|_{\psi_0}$$
$$\frac{\rho_0}{\bar{\rho}} = \frac{\psi_0^3}{3\Lambda}$$
$$\alpha = \sqrt{\frac{a(n+1)}{4\pi G}\rho_0^{\frac{1}{n}-1}}$$

n	0	1	2	3	4	5
ψ_0	$\sqrt{6}$	π	4.35	6.89	14.97	∞
$-\psi_0^2 \left.\frac{d\theta}{d\psi}\right\|_{\psi_0}$	$2\sqrt{6}$	π	2.41	2.02	1.80	1.7
$\frac{\rho_0}{\bar{\rho}}$	1	$\frac{1}{3}\pi^2$	11.40	54.16	621.92	∞

n	1.5	3
ψ_0	3.65	6.90
$-\psi_0^2 \left.\frac{d\theta}{d\psi}\right\|_{\psi_0}$	2.71	2.02
$\frac{\rho_0}{\bar{\rho}}$	5.99	54.16

$$R^2 = \alpha^2\psi_0^2$$
$$R^2 = \frac{a(n+1)}{4\pi G}\rho_0^{\frac{1}{n}-1}\psi_0^2$$
$$R^2 = \frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\frac{M}{\frac{4}{3}\pi R^3}\right)^{\frac{1}{n}-1}\psi_0^2$$
$$R^{2+\frac{3}{n}-3} = \frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{n}}\psi_0^2 M^{\frac{1}{n}-1} \quad *$$
$$R = \left(\frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{n}}\psi_0^2\right)^{\frac{1}{\frac{3}{n}-1}} M^{\frac{\frac{1}{n}-1}{\frac{3}{n}-1}}$$
$$\therefore R = \left(\frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{n}}\psi_0^2\right)^{\frac{n}{3-n}} M^{\frac{n-1}{n-3}}$$

Radius vs mass relationship for a polytropic star

This is *undefined* for $n = 3$

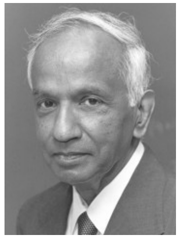
Careful! a may vary with M and R, so not a particularly useful equation in itself …

* If $n = 3$ then fixed mass given polytropic constant a

$$M = \left(\frac{a(3+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{3}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{3}}\psi_0^2\right)^{\frac{3}{3-1}}$$
$$M = \left(\pi^{-1+\frac{2}{3}} \times 2^{\frac{4}{3}} \times 3^{\frac{2}{3}-\frac{2}{3}}\frac{a\Lambda^{\frac{2}{3}}}{G}\right)^{\frac{3}{2}}$$
$$M = \left(\pi^{-\frac{1}{3}} \times 2^{\frac{4}{3}}\frac{a\Lambda^{\frac{2}{3}}}{G}\right)^{\frac{3}{2}}$$
$$M = \left(\pi^{-\frac{1}{2}} \times 2^2\right)\Lambda\left(\frac{a}{G}\right)^{\frac{3}{2}}$$
$$\therefore M = \frac{4\Lambda}{\sqrt{\pi}}\left(\frac{a}{G}\right)^{\frac{3}{2}}$$

This gives the **Chandrasekhar mass limit** for white dwarfs (calculate a from a model of *degeneracy pressure*)

$$M = \frac{\Lambda\sqrt{3}}{\pi\sqrt{32}}\left(\frac{hc}{G\mu^{\frac{4}{3}}}\right)^{\frac{3}{2}} \approx 1.44M_\odot$$



Subrahmanyan Chandrasekhar (1910-1995)

For a MS star model (ignoring radiation pressure)

- **Inputs** are **star mass, hydrogen and helium mass fractions** and **polytropic index.**
- Get effective temperature and luminosity from Hertzsprung-Russell MS correlations with mass.
- Find radius from luminosity and effective temperature
- Hence determine average density, and therefore core density from Lane-Emden condensation, given polytropic index.
- Calculate core temperature assuming ideal gas model
- Calculate radial extent and then mass enclosed and density vs radial extent from Lane-Emden.
- Calculate pressure from ideal gas and polytropic model.
- Calculate temperature using ideal gas equation.
- Then calculate luminosity and convective stability using radiative and convective transport models, and model of energy production (and opacity) via nuclear fusion.

Example on next page!

Numeric method for calculating luminosity

Idea is to add up contributions from a set of concentric shells, then scale the final luminosity so that it equals L, which we computed from the empirical relationship.

```
N = length(s.r); l = zeros(1,N);
for n=1:(N-1)
    %Shell width (in m)
    dr = s.r(n+1)-s.r(n);
    %Mass of shell of width dr (in kg)
    dm = 4*pi*( s.r(n)^2 )*s.rho(n)*dr;
    %Contribution to luminosity (in W) using nuclear fusion power model
    yes = fusion_yes_or_no( s.T(n), s.fusion_model);
    if yes==1;
        %Temperature is hot enough for nuclear fusion
        dl = dm * ( X_H^AX )*( X_He^AY )*( (1-X_H-X_He)^AZ )*( s.rho(n)^B )*( s.T(n)^C );
    else
        dl = 0;
    end
    %Cumulatively sum luminosity
    l(n+1) = l(n) + dl;
end
%Scale such that luminosity at surface is L
s.l = s.L*l/l(N);
```

δr

$\delta m = 4\pi r^2 \rho \delta r$

$\delta l = \delta m \times X^{A_H} Y^{A_{He}} Z^{A_{metal}} \rho^{\beta} T^{\chi}$

$l_{n+1} = l_n + \delta l$

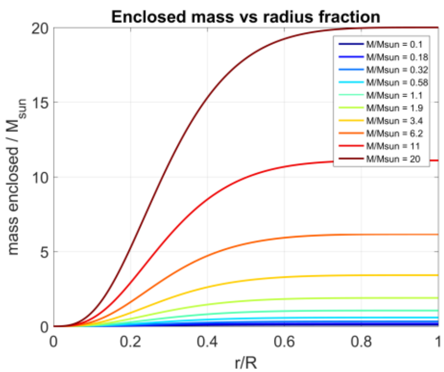
Use the same temperature thresholds

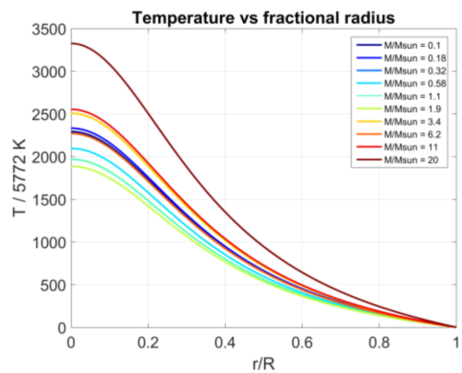
```
fusion_model = 'pp';
AX = 2; AY = 0; AZ=0; B = 1; C = 4;
if ( T0>1.5e7 ) & (T0<1e8)
    %CNO cycle
    fusion_model = 'CNO';
    AX = 1; AY = 0; AZ=1; B = 1; C = 17;
elseif T0>1e8
    %Triple-alpha
    fusion_model = 'Triple-alpha';
    AX = 0; AY = 3; AZ=0; B = 2; C = 40;
end
```

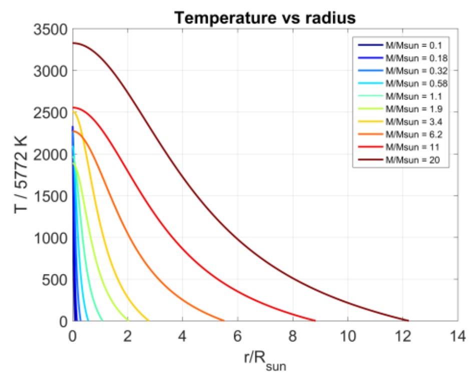
e.g. pp: $\varepsilon = \eta_{pp} X^2 \rho T^4$

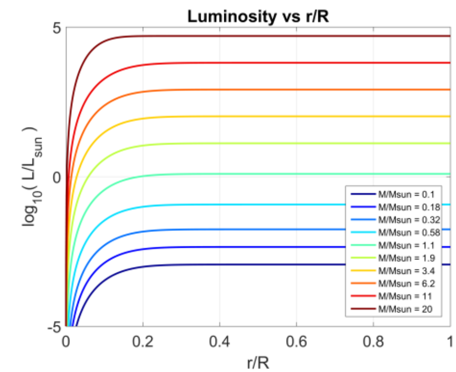
$A_H = 2, A_{He} = 0, A_{metal} = 0, \beta = 1, \gamma = 4$

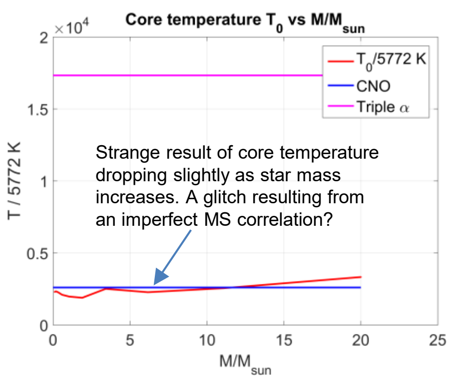
Model run using masses between 0.1 and 20 solar masses. $X = 0.747, Y = 0.236, Z = 0.017 \quad \therefore \mu \approx 0.6 m_p$ i.e. same as the Sun.

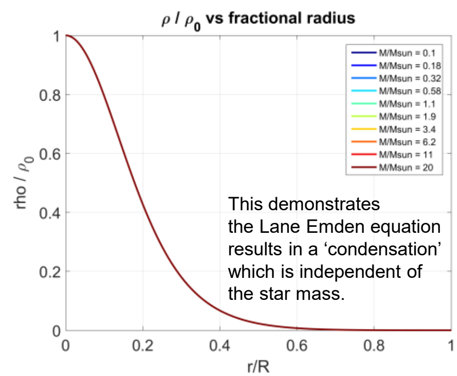


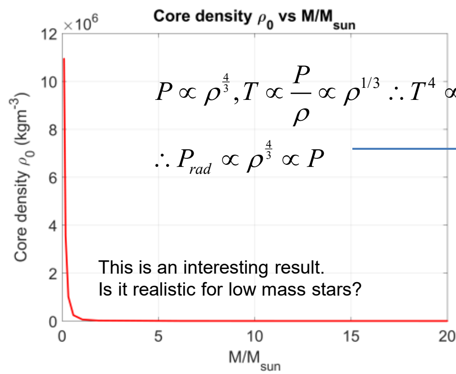






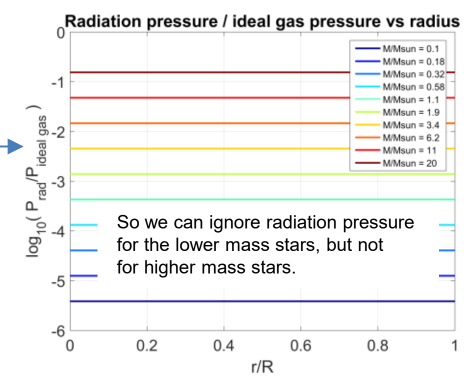






$P \propto \rho^{\frac{4}{3}}, T \propto \frac{P}{\rho} \propto \rho^{1/3} \therefore T^4 \propto \rho^{4/3}$

$\therefore P_{rad} \propto \rho^{\frac{4}{3}} \propto P$



Luminosity, temperature gradient relationship for radiative heat transport, and opacity

Opacity is defined by:

$$\kappa = -\frac{A \times \frac{1}{\Phi} d\Phi}{A \rho dr}$$

$$\Rightarrow \frac{d\Phi}{\Phi} = -\kappa \rho dr$$

$$\Rightarrow \Phi = \Phi_0 e^{-\kappa \rho r}$$ If density and opacity constant

i.e. fractional radiation flux (W/m²) scattered by matter area A and thickness dr, per kg of this matter, multiplied by area A.

Or a measure of the attenuation of radiation, Since when:

$$r = \frac{1}{\kappa\rho},\ \Phi = \frac{\Phi_0}{e} \approx 0.37\Phi_0.$$

The radiative flux (i.e. power per square metre) through a radial element of thickness dr is

$$d\Phi = d\left(\sigma T^4\right) = -\kappa\rho dr \times \frac{l}{4\pi r^2}$$

$$\therefore 4\sigma T^3 dT = -\frac{\kappa\rho l dr}{4\pi r^2}$$

$$\therefore \frac{dT}{dr} = -\frac{\kappa\rho l}{16\pi r^2 \sigma T^3}$$

Turns out this is not quite correct, as one should integrate over all angles. The correct version of the **Eddington Equation for radiative transfer** is:

$$\therefore \frac{dT}{dr} = -\frac{3}{4}\frac{\kappa\rho l}{16\pi r^2 \sigma T^3}$$

$$\kappa = -\frac{4}{3}\frac{16\pi r^2 \sigma T^3}{\rho l}\frac{dT}{dr}$$

i.e. assuming temperature gradient is *only* due to radiative transport. This is thought not to be a good model near the star radius.

Arthur Eddington (1882-1944)

Note we know T and P vs r from Lane-Emden Model so we can compute the temperature gradient

Convective stability

The **Schwarzschild criterion** for convective instability (i.e. convection is likely to occur) is:

Radiative process | Adiabatic process

$$\left|\frac{dT}{dr}\right|_{rad} > \left|\frac{dT}{dr}\right|_{ad}$$

i.e. no heat is transferred

For an **adiabatic process** involving an ideal gas: $V \propto \frac{T}{P}$

$$d(PV^\gamma) = 0$$

$$V \propto \frac{T}{P} \Rightarrow PV^\gamma \propto P^{1-\gamma}T^\gamma$$

$$\therefore d(PV^\gamma) = 0 \Rightarrow P^{1-\gamma}\gamma T^{\gamma-1}dT + T^\gamma(1-\gamma)P^{-\gamma}dP = 0$$

$$\therefore dT = \frac{\gamma-1}{\gamma}\frac{T^\gamma P^{-\gamma}}{P^{1-\gamma}T^{\gamma-1}}dP = \frac{\gamma-1}{\gamma}\frac{dP}{PT^{-1}}$$

$$\therefore \frac{dT}{dr} = \frac{\gamma-1}{\gamma}\frac{T}{P}\frac{dP}{dr}$$

$$\gamma = \frac{c_p}{c_v} = \frac{5}{3}$$

for a fully ionized ideal gas

So convection in a star possible if:

$$\frac{3}{4}\frac{\kappa\rho l}{16\pi r^2\sigma T^3} > \frac{\gamma-1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right|$$

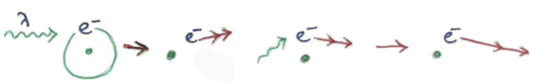
But if ionization is partial then $\gamma \rightarrow 1$

Models of opacity

But what is the constant?!

Kramer's opacity $\kappa \approx \kappa_0 \rho T^{-3.5}$

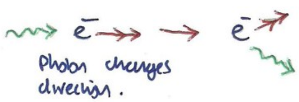
'Bound-free' and 'free-free' scattering
i.e. electron absorbs photon and is scattered.
Appropriate in low temperature, low density stars and radiative atmospheres.

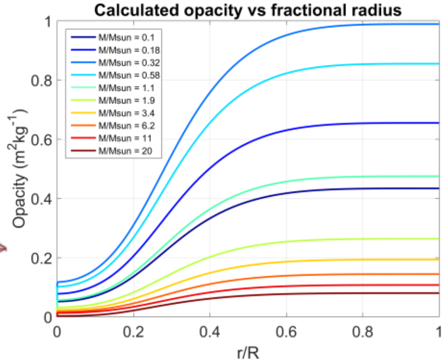


$$\kappa \approx 0.02(1+X)\ \mathrm{m^2kg^{-1}}$$

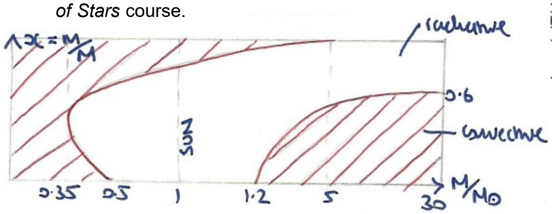
X is Hydrogen mass fraction of star

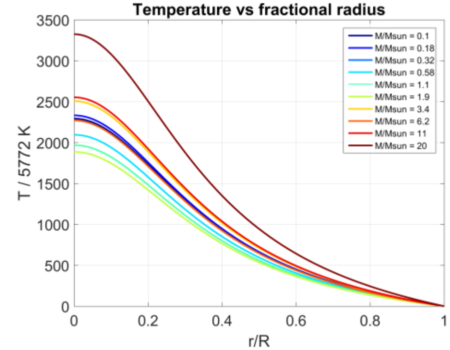
Low energy limit of **Compton scattering**. i.e. electron is scattered by photon, but photon changed in wavelength and scattered
Occurs in cores of most stars and atmospheres of hot stars.

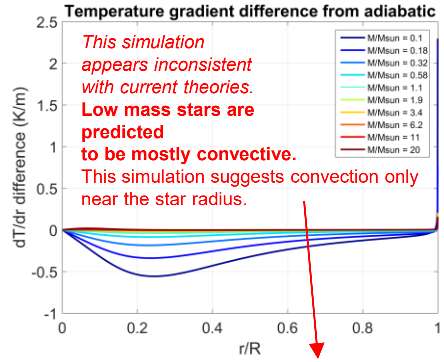




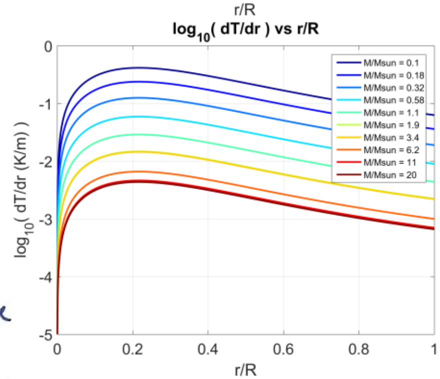
This figure is from by Uni notes from the Cambridge Part III *Structure and Evolution of Stars* course.



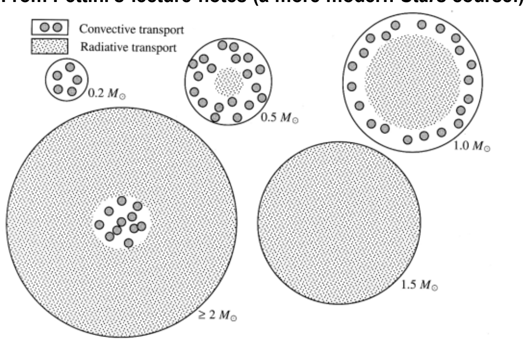


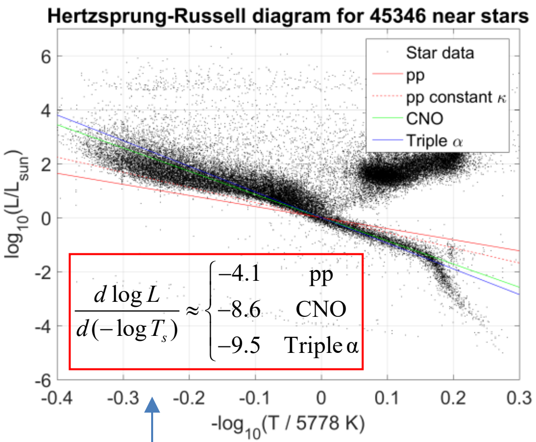


This simulation appears inconsistent with current theories.
Low mass stars are predicted to be mostly convective.
This simulation suggests convection only near the star radius.



From Pettini's lecture notes (a more modern *Stars* course!):

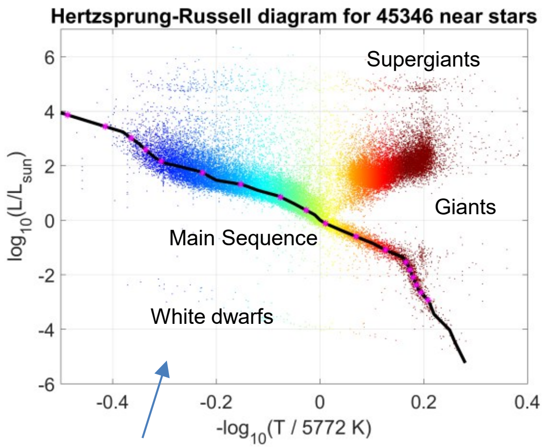




$$\frac{d\log L}{d(-\log T_s)} \approx \begin{cases} -4.1 & \text{pp} \\ -8.6 & \text{CNO} \\ -9.5 & \text{Triple}\ \alpha \end{cases}$$

Gradients of HR diagram suggested by **homology relationships**. The sun is at (0,0).

See later!



Stars chosen for the simulations described are indicated by the magenta stars

Hertzsprung-Russell diagram, with star data colour coded by colour index for star

```
%Set colour index to have max and min values
Which matches the blue to red colour scale
B_minus_V = stars.colour_index_B_minus_V;
B_minus_V( B_minus_V > 1.4) = 1.4;
B_minus_V( B_minus_V < -0.33) = -0.33;

%Calculate star effective temperature (K)
using
Ballesteros' formula
%https://en.wikipedia.org/wiki/Color_index
```

$$T = 4600\text{K} \times \left(\frac{1}{0.92(B-V)+1.7} + \frac{1}{0.92(B-V)+0.62} \right)$$

Let's run the **Lane-Emden model for the Sun**, to enable the core temperature to be estimated.

$$M_\odot = 1.989 \times 10^{30}\ \text{kg}$$
$$R_\odot = 696{,}340\ \text{km}$$
$$L_\odot = 3.846 \times 10^{26}\ \text{Js}^{-1}$$
$$T_\odot = 5778\ \text{K}$$

$$X = 0.747, Y = 0.236, Z = 0.017$$
$$\therefore \mu \approx 0.6 m_p$$

$$\Lambda = -\psi_0^2 \left.\frac{d\theta}{d\psi}\right|_{\psi_0} \quad M = 4\pi\alpha^3 \rho_0 \Lambda \quad R = \alpha\psi_0$$

$$\therefore \bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{4\pi\alpha^3\rho_0\Lambda}{\frac{4}{3}\pi\alpha^3\psi_0^3} = \frac{3\rho_0\Lambda}{\psi_0^3}$$

$$\therefore \frac{\rho_0}{\bar{\rho}} = \frac{\psi_0^3}{3\Lambda}$$

$$R = 6.89\alpha \ \therefore \alpha = \frac{R_\odot}{6.89}$$

$$\alpha = \sqrt{\frac{RT_0(3+1)}{4\pi G\mu\rho_0}}$$ Assuming ideal gas in core

$$\therefore \frac{4k_B T_0}{4\pi G\mu\rho_0} = \left(\frac{R_\odot}{6.89}\right)^2$$

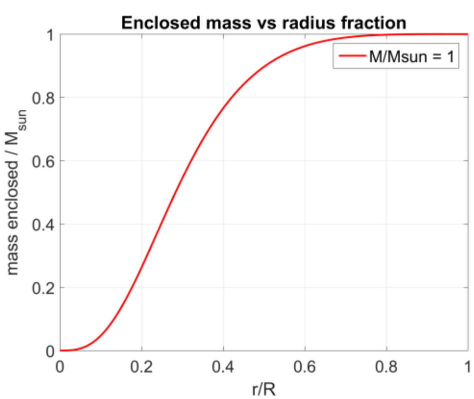
$$\therefore T_0 = \frac{\pi G\mu\rho_0}{k_B}\left(\frac{R_\odot}{6.89}\right)^2$$

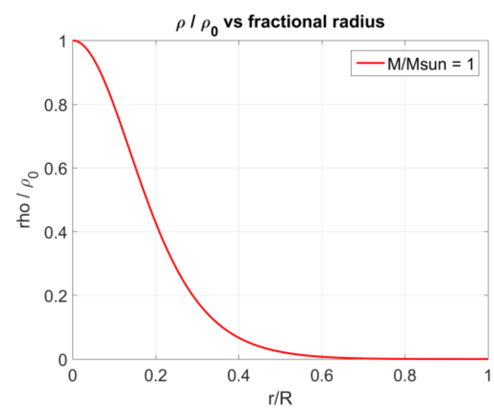
$$\rho_0 = 53.95 \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}$$

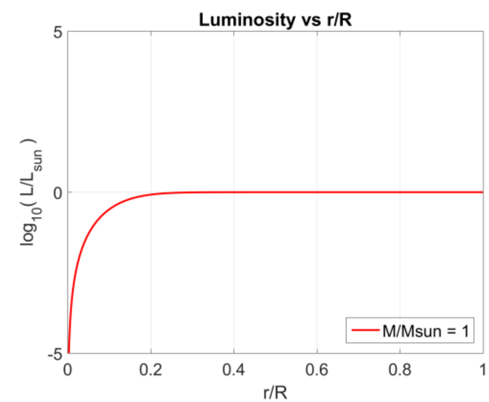
$$\therefore T_0 = \frac{\pi G\mu}{k_B}\left(\frac{R_\odot}{6.89}\right)^2 \times 53.95 \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}$$

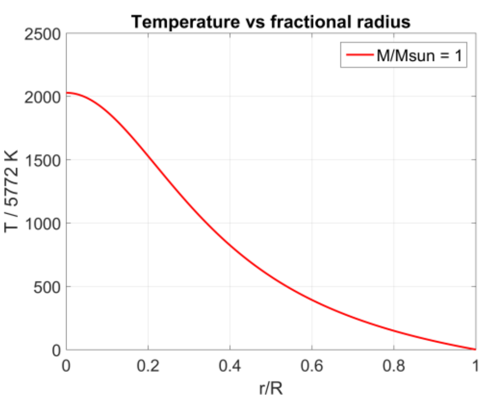
$$\therefore T_0 = \frac{\pi G\mu}{k_B}\left(\frac{R_\odot}{6.89}\right)^2 \times 53.95 \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}$$

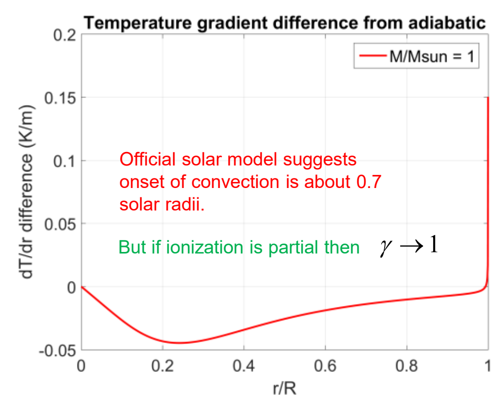
$$\therefore T_0 = \left(\frac{3 \times 53.95}{4 \times 6.89^2}\right)\frac{GM_\odot\mu}{k_B R_\odot} \approx 1.18 \times 10^7\ \text{K}$$











Star structure equations in Lagrangian coordinates

(i.e. in terms of m)

Let's assume radiation-dominated heat transport, but where radiation pressure can be ignored.

$$\frac{dr}{dm}=\frac{1}{4\pi r^2\rho}$$

$$\frac{dl}{dm}=\varepsilon=\eta\rho^\alpha T^\beta$$

$$\frac{dT}{dm}=-\frac{3\kappa l}{256\pi^2 r^4\sigma T^3}$$

$$\frac{dr}{dm}=\frac{1}{4\pi r^2\rho}$$

$$\frac{dP}{dm}=\frac{dP}{dr}\times\frac{dr}{dm}=-\frac{Gm}{4\pi r^4}$$

$$\frac{dl}{dm}=\frac{dl}{dr}\times\frac{dr}{dm}=\varepsilon$$

$$P=\frac{\rho k_B T}{\mu}+\frac{4\sigma}{3c}T^4,\quad \frac{dP}{dm}=-\frac{Gm}{4\pi r^4}$$

$$\therefore \frac{k_B T}{\mu}\frac{d\rho}{dm}+\frac{\rho k_B}{\mu}\frac{dT}{dm}+\frac{16\sigma}{3c}T^3\frac{dT}{dm}=-\frac{Gm}{4\pi r^4}$$

$$\therefore \frac{d\rho}{dm}=-\frac{\mu}{k_B T}\frac{Gm}{4\pi r^4}-\frac{\mu}{RT}\frac{dT}{dm}\left(\frac{\rho k_B}{\mu}+\frac{16\sigma}{3c}T^3\right)$$

$$\therefore \frac{d\rho}{dm}=-\frac{\mu}{k_B T}\frac{Gm}{4\pi r^4}-\frac{dT}{dm}\left(\frac{\rho}{T}+\frac{16\mu\sigma}{3k_B c}T^2\right)$$

Boundary conditions are: $m=M,\ r=R,\ T=T_s$; $L=l(M)=4\pi R^2\sigma T_e^4$

Idea is to guess initial conditions in order to arrive at the boundary conditions which match observables such as:

$$M_\odot=1.989\times10^{30}\ \text{kg}$$
$$R_\odot=696,340\ \text{km}$$
$$L_\odot=3.846\times10^{26}\ \text{Js}^{-1}$$
$$T_\odot=5778\ \text{K}$$

Idea is to work backwards from the surface, and simultaneously from core to surface, changing the parameters until the solutions converge. This is the method developed by M. Schwarzchild in 1958.

Initial conditions to guess are:

$T_0,\ \rho_0,\ \eta,\ \kappa_0$

Might be able to reduce the number?
For continuity of opacity, assuming low energy Compton scattering occurs in the cores of most stars:

$$0.02(1+X)\approx\kappa_0\rho_0 T_0^{-3.5}$$

$$\Rightarrow \kappa_0=\frac{0.02(1+X)}{\rho_0 T_0^{-3.5}}$$

This is what we did in the Lane Emden model

… or could assign at a lower T beyond the core?

A sensible pre-solver step is to scale the equations, so the coupled non-linear differential equations are in terms of **dimensionless variables**

$$x=m/M,\ M\rightarrow MM_\odot\ \ \therefore m\rightarrow xMM_\odot$$

$$l\rightarrow lL_\odot,\ T\rightarrow TT_\odot,\ r\rightarrow rR_\odot,\ \rho\rightarrow\rho\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}$$

$0\le x\le 1$ and fix M for a given solver.

Mass in shells

$$\frac{dr}{dm}=\frac{1}{4\pi r^2\rho}$$

$$\therefore \frac{R_\odot}{MM_\odot}\frac{dr}{dx}=\frac{1}{4\pi r^2\rho}\frac{1}{R_\odot^2}\frac{\frac{4}{3}\pi R_\odot^3}{M_\odot}$$

$$\therefore \frac{dr}{dx}=\frac{M}{3r^2\rho}$$

Energy generation by nuclear fusion

$$\frac{dl}{dm}=\eta\rho^\alpha T^\beta$$

$$\therefore \frac{L_\odot}{MM_\odot}\frac{dl}{dx}=\eta\rho^\alpha\left(\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}\right)^\alpha T_\odot^\beta T^\beta$$

$$\therefore \frac{dl}{dx}=\Omega M\rho^\alpha T^\beta,\quad \Omega=\eta\frac{M_\odot}{L_\odot}\left(\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}\right)^\alpha T_\odot^\beta$$

Convective (adiabatic) energy transport

$$\frac{dT}{dm}=-\tfrac{1}{4\pi}\left(1-\frac{1}{\gamma}\right)\frac{\mu}{\rho k_B}\frac{Gm}{r^4}$$

$$\therefore \frac{T_\odot}{MM_\odot}\frac{dT}{dx}=-\tfrac{1}{4\pi}\left(1-\frac{1}{\gamma}\right)\frac{\mu}{\rho k_B}\frac{Gx\,MM_\odot}{r^4\,R_\odot^4}\frac{1}{\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}}$$

$$\therefore \frac{dT}{dx}=-\Xi\frac{M^2 x}{\rho r^4},\quad \Xi=\frac{\mu G}{3k_B}\left(1-\frac{1}{\gamma}\right)\frac{M_\odot}{R_\odot T_\odot}$$

Radiative energy transport

$$\frac{dT}{dm}=-\frac{3\kappa_0\rho^y T^{-z}l}{256\pi^2 r^4\sigma T^3}$$

$$\kappa=\kappa_0\rho^y T^{-z}$$

Radiative high T $\ y=0,z=0$

Radiative low T $\ y=1,z=3.5$

$$\therefore \frac{T_\odot}{MM_\odot}\frac{dT}{dx}=-\frac{3\kappa_0 l\rho^y T^{-z}}{256\pi^2 r^4\sigma T^3}\left(\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}\right)^y T_\odot^{-z}\frac{L_\odot}{R_\odot^4 T_\odot^3}$$

$$\therefore \frac{dT}{dx}=-\Theta\frac{Ml\rho^y}{r^4T^{z+3}},\quad \Theta=\frac{3\kappa_0}{256\pi^2\sigma}\left(\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}\right)^y\frac{L_\odot M_\odot}{R_\odot^4 T_\odot^{z+4}}$$

Hydrostatic equilibrium

$$\frac{d\rho}{dm}=-\frac{\mu}{RT}\frac{Gm}{4\pi r^4}-\frac{dT}{dm}\left(\frac{\rho}{T}+\frac{16\mu\sigma}{3k_B c}T^2\right)$$

$$\therefore \frac{\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}}{MM_\odot}\frac{d\rho}{dx}=-\frac{\mu}{k_B TT_\odot}\frac{GxMM_\odot}{4\pi r^4R_\odot^4}-\frac{T_\odot}{MM_\odot}\frac{dT}{dx}\left(\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3 T_\odot}\frac{\rho}{T}+\frac{16\mu\sigma}{3k_B c}T^2T_\odot^2\right)$$

$$\therefore \frac{d\rho}{dx}=-M\tfrac{4}{3}\pi R_\odot^3\left\{\left(\frac{\mu}{k_B T_\odot}\frac{GM_\odot}{4\pi R_\odot^4}\right)\frac{xM}{r^4T}+\frac{1}{M}\frac{dT}{dx}\left(\frac{1}{\frac{4}{3}\pi R_\odot^3}\frac{\rho}{T}+\frac{16\mu\sigma T_\odot^3}{3k_B cM_\odot}T^2\right)\right\}$$

$$\therefore \frac{d\rho}{dx}=-\left\{\left(\frac{\mu}{k_B T_\odot}\frac{3GM_\odot}{R_\odot}\right)\frac{xM^2}{r^4T}+\frac{dT}{dx}\left(\frac{\rho}{T}+\frac{\frac{4}{3}\pi R_\odot^3\,16\mu\sigma T_\odot^3}{3k_B cM_\odot}T^2\right)\right\}$$

$$\therefore \frac{d\rho}{dx}=-\omega\frac{xM^2}{r^4T}-\frac{dT}{dx}\left(\frac{\rho}{T}+\lambda T^2\right)$$

$$\omega=\frac{\mu}{k_B T_\odot}\frac{3GM_\odot}{R_\odot},\quad \lambda=\frac{64\pi R_\odot^3\mu\sigma T_\odot^3}{9k_B cM_\odot}$$

The **Schwarzschild criterion** for convective instability (i.e. convection is likely to occur) is:

$$\left|\frac{dT}{dr}\right|_{rad}>\left|\frac{dT}{dr}\right|_{ad}\Rightarrow\left|\frac{dT}{dx}\frac{dx}{dr}\right|_{rad}>\left|\frac{dT}{dx}\frac{dx}{dr}\right|_{ad}$$

$$\Rightarrow\Theta\frac{Ml\rho^y}{r^4T^{z+3}}>\Xi\frac{M^2x}{\rho r^4}\Rightarrow\Theta\frac{l\rho^{y+1}}{T^{z+3}}>\Xi Mx$$

A homological approach

Let's further scale our (now dimensionless) variables l, r, T, ρ to be further scaled by powers of M. Can we write each equation of stellar structure in such a way that they are independent of M and hence appropriate for all stars (in the Main Sequence) ?

$$l \to M^A l,\ r \to M^B r,\ T \to M^C T,\ \rho \to M^D \rho$$

$$\frac{dr}{dx} = \frac{M}{3r^2\rho}$$

$$\therefore M^B \frac{dr}{dx} = \frac{M}{3r^2\rho} M^{-2B-D}$$

$$\therefore \frac{dr}{dx} = \frac{1}{3r^2\rho}$$

$$\Rightarrow B = 1 - 2B - D$$

$$\Rightarrow 3B + D = 1$$

$$\frac{dl}{dx} = \Omega M \rho^\alpha T^\beta, \quad \Omega = \eta \frac{M_\odot}{L_\odot}\left(\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}\right)^\alpha T_\odot^\beta$$

$$\therefore M^A \frac{dl}{dx} = \Omega M \rho^\alpha T^\beta M^{\alpha D + \beta C}$$

$$\therefore \frac{dl}{dx} = \Omega \rho^\alpha T^\beta$$

$$\Rightarrow A = 1 + \alpha D + \beta C$$

$$\Rightarrow A - \alpha D - \beta C = 1$$

pp:	$\alpha = 1, \beta = 4$
CNO:	$\alpha = 1, \beta = 17$
Triple-alpha:	$\alpha = 2, \beta = 40$

$$\frac{d\rho}{dx} = -\omega \frac{xM^2}{r^4 T} - \frac{dT}{dx}\left(\frac{\rho}{T} + \lambda T^2\right)$$

$$\omega = \frac{\mu}{RT_\odot}\frac{3GM_\odot}{R_\odot}, \quad \lambda = \frac{64\pi R_\odot^3 \mu \sigma T_\odot^3}{9RcM_\odot}$$

$$\therefore M^D \frac{d\rho}{dx} = -\omega \frac{xM^2}{r^4 T} M^{-4B-C} - M^C \frac{dT}{dx}\left(M^{D-C}\frac{\rho}{T} + \lambda M^{2C} T^2\right)$$

$$\therefore \frac{d\rho}{dx} = -\frac{\omega x}{r^4 T} - \frac{dT}{dx}\left(\frac{\rho}{T} + \lambda T^2\right)$$

$$\Rightarrow D = 2 - 4B - C \Rightarrow D + 4B + C = 2$$

AND $D = 3C$

Ignore this extra constraint **if can ignore radiation pressure**. Otherwise this may invalidate the homology argument. $\lambda \ll 1$

Radiative

$$\frac{dT}{dx} = -\Theta \frac{Ml\rho^y}{r^4 T^{z+3}}, \quad \Theta = \frac{3\kappa_0}{256\pi^2\sigma}\left(\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}\right)^y \frac{L_\odot M_\odot}{R_\odot^4 T_\odot^{z+3}}$$

$$\therefore M^C \frac{dT}{dx} = -\Theta \frac{Ml\rho^y}{r^4 T^{z+3}} M^{A+yD-(z+3)C-4B}$$

$$\therefore \frac{dT}{dx} = -\frac{\Theta l \rho^y}{r^4 T^{z+3}}$$

$$\Rightarrow C = 1 + A + yD - (z+3)C - 4B$$

$$\Rightarrow -A + 4B + (z+4)C - yD = 1$$

Radiative high T $\quad y = 0, z = 0$

Radiative low T $\quad y = 1, z = 3.5$

Kramer's opacity

$$\therefore \begin{pmatrix} 0 & 3 & 0 & 1 \\ 1 & 0 & -\beta & -\alpha \\ 0 & 4 & 1 & 1 \\ -1 & 4 & z+4 & -y \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 1 \\ 1 & 0 & -\beta & -\alpha \\ 0 & 4 & 1 & 1 \\ -1 & 4 & z+4 & -y \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

Use a computer programme to invert this matrix e.g. MATLAB.

We can now predict the *gradients* of the HR diagram!

Low mass, low temperature star. pp chain fusion.
$\alpha = 1, \beta = 4, y = 1, z = 3.5$

$$L \propto M^{5.5},\ R \propto M^{0.077},\ T_0 \propto M^{0.92},\ \rho_0 \propto M^{0.77}$$

High mass, high temperature star. CNO fusion.
$\alpha = 1,\ \beta = 17,\ y = 0,\ z = 0$

$$L \propto M^3,\ R \propto M^{0.8},\ T_0 \propto M^{0.2},\ \rho_0 \propto M^{-1.4}$$

High mass, higher temperature star. Triple alpha fusion.
$\alpha = 2,\ \beta = 40,\ y = 0,\ z = 0$

$$L \propto M^3,\ R \propto M^{0.87},\ T_0 \propto M^{0.13},\ \rho_0 \propto M^{-1.6}$$

Non-convective criterion

$$\frac{dT}{dx} < -\Xi \frac{M^2 x}{\rho r^4}, \quad \Xi = \frac{\mu G}{3R}\left(1 - \frac{1}{\gamma}\right)\frac{M_\odot}{R_\odot T_\odot}$$

$$\therefore M^C \frac{dT}{dx} < -\Xi \frac{M^2 x}{\rho r^4} M^{-D-4B}$$

$$\therefore \frac{dT}{dx} < -\frac{\Xi x}{\rho r^4}$$

$$\Rightarrow C = 2 - D - 4B$$

$$\Rightarrow C + D + 4B = 2$$

This yields the same equation for powers of M as the density equation, so this is consistent with our homological approach.

Applying homology relations :

$$L = l_{x=1} M^A,\ R = r_{x=1} M^B,$$

$$\log L = \log l_{x=1} + A \log M, \quad \log R = \log r_{x=1} + B \log M$$

Use (sun scaled) luminosity vs radius and effective temperature relationship:

e.g $L \to \frac{L}{L_\odot}$

$$L = R^2 T_e^4 \Rightarrow T_s = \left(\frac{L}{R^2}\right)^{\frac{1}{4}}$$

$$\therefore T_e = \left(\frac{l_{x=1} M^A}{r_{x=1} M^{2B}}\right)^{\frac{1}{4}} = \left(\frac{l_{x=1}}{r_{x=1}}\right)^{\frac{1}{4}} M^{\frac{A-2B}{4}}$$

$$\therefore -\log T_e = -\tfrac{1}{4}\log\left(\frac{l_{x=1}}{r_{x=1}}\right) - \left(\frac{A-2B}{4}\right)\log M$$

$$\log L = \log l_{x=1} + A \log M$$

$$\therefore \frac{d \log L}{d(-\log T_e)} = \frac{d \log L}{d \log M} \times \frac{d \log M}{d(-\log T_e)} = -\frac{4A}{A-2B} = \frac{-4}{1-2B/A}$$

HR diagram predicted gradient

$$\frac{d \log L}{d(-\log T_e)} \approx \begin{cases} -4.1 & \text{pp} \\ -8.6 & \text{CNO} \\ -9.5 & \text{Triple}\,\alpha \end{cases}$$

Eddington stellar model

In the Eddington model, define a parameter which sets the ratio between gas and radiation + gas pressure, and **assume this is a constant throughout the star.**

$$P = \frac{\rho k_B T}{\mu} + \frac{4\sigma}{3c}T^4$$

Ideal gas

Radiation pressure

$$P_g = \beta P = \beta\left(P_g + P_r\right) \Rightarrow P_g(1-\beta) = \beta P_r$$

$$\therefore \frac{\rho k_B T}{\mu}(1-\beta) = \beta\frac{4\sigma}{3c}T^4$$

$$\Rightarrow T = \left(\frac{3k_B c}{4\sigma\mu}\frac{1-\beta}{\beta}\right)^{\frac{1}{3}}\rho^{\frac{1}{3}}$$

$M_\odot = 1.988\times10^{30}$ kg
$k_B = 1.381\times10^{-23}$ JK^{-1}
$G = 6.674\times10^{-11}$ m^{3}kg^{-1}s^{-2}
$\sigma = 5.670\times10^{-8}$ Wm^{-2}K^{-4}
$m_p = 1.673\times10^{-27}$ kg
$c = 2.998\times10^{8}$ ms^{-1}

$$T = \left(\frac{3k_B c}{4\sigma\mu}\frac{1-\beta}{\beta}\right)^{\frac{1}{3}}\rho^{\frac{1}{3}}$$

$$P = \frac{P_g}{\beta}, \quad P_g = \frac{\rho k_B T}{\mu}$$ Ideal gas

$$\therefore P = \frac{\rho k_B}{\beta\mu}\left(\frac{3k_B c}{4\sigma\mu}\frac{1-\beta}{\beta}\right)^{\frac{1}{3}}\rho^{\frac{1}{3}}$$

$$\therefore P = \left(\frac{3k_B^4 c}{4\sigma\mu^4}\frac{1-\beta}{\beta^4}\right)^{\frac{1}{3}}\rho^{1+\frac{1}{3}}$$

i.e. a polytrope of order three

For polytrope: $P = a\rho^{1+\frac{1}{3}}$ $\quad \therefore a = \left(\frac{3k_B c}{4\sigma\mu}\frac{1-\beta}{\beta^4}\right)^{\frac{1}{3}}$

$$\Rightarrow M = \left(\frac{a(3+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{3}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{3}}\psi_0^2\right)^{\frac{3}{3-1}}$$

using Lane-Emden model for $n = 3$ polytropes

$\psi_0 = 6.90$
$\Lambda = 2.02$

Arthur Eddington
(1882-1944)

$$M = \left(\frac{a(3+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{3}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{3}}\psi_0^2\right)^{\frac{3}{3-1}}$$

$$a = \left(\frac{3k_B^4 c}{4\sigma\mu^4}\frac{1-\beta}{\beta}\right)^{\frac{1}{3}}$$

$$\therefore M = \left(\frac{\left(\frac{4}{3}\pi\right)^{\frac{2}{3}}}{\pi G}\left(\frac{3k_B^4 c}{4\sigma\mu^4}\right)^{\frac{1}{3}}\left(\frac{\psi_0^3}{3\Lambda}\right)^{-\frac{2}{3}}\psi_0^2\right)^{\frac{3}{2}}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}}$$

$$\therefore M = \left(\pi^{\frac{2}{3}-1}\times 3^{-\frac{2}{3}+\frac{1}{3}+\frac{2}{3}}\times 2^{\frac{4}{3}-\frac{2}{3}}\left(\frac{k_B^4 c}{\sigma\mu^4 G^3}\right)^{\frac{1}{3}}\Lambda^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}}$$

$$\therefore M = \left(\pi^{-\frac{1}{3}}\times 3^{\frac{1}{3}}\times 2^{\frac{2}{3}}\left(\frac{k_B^4 c}{\sigma\mu^4 G^3}\right)^{\frac{1}{3}}\Lambda^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}}$$

$$\therefore M = \left(\pi^{-\frac{1}{2}}\times 3^{\frac{1}{2}}\times 2\left(\frac{k_B^4 c}{\sigma\mu^4 G^3}\right)^{\frac{1}{2}}\Lambda\right)\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}}$$

$$\therefore M = \left(\pi^{-\frac{1}{2}}\times 3^{\frac{1}{2}}\times 2\left(\frac{k_B^4 c}{\sigma\mu^4 G^3}\right)^{\frac{1}{2}}\Lambda\right)\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}}$$

$$\therefore \frac{M}{M_\odot} = \sqrt{\tfrac{12}{\pi}\frac{\Lambda^2 k_B^4 c}{\sigma G^3\mu^4 M_\odot^2}}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}}$$

$$\therefore \frac{M}{M_\odot} \approx \frac{18.0}{\left(\frac{\mu}{m_p}\right)^2}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}}$$

But setting β to zero *doesn't* yield a finite upper limit for star mass, as this would imply an infinite mass.

Solar parameters
$X = 0.747, Y = 0.236$
$\therefore \mu \approx 0.6m_p$

$$\frac{\mu}{m_p} = \frac{4}{6X+Y+2}$$

$M = M_\odot$
$\beta = 0.9996$

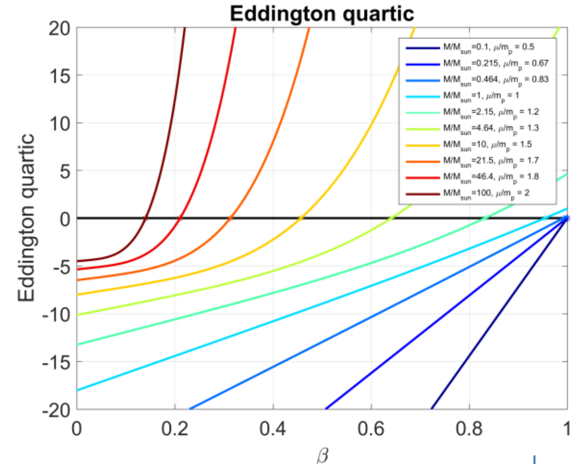
So radiation pressure not so important for stars of small multiples of solar masses.

Note if one knows the star mass then this yields **Eddington's Quartic** for the fraction β of total pressure that is gas pressure

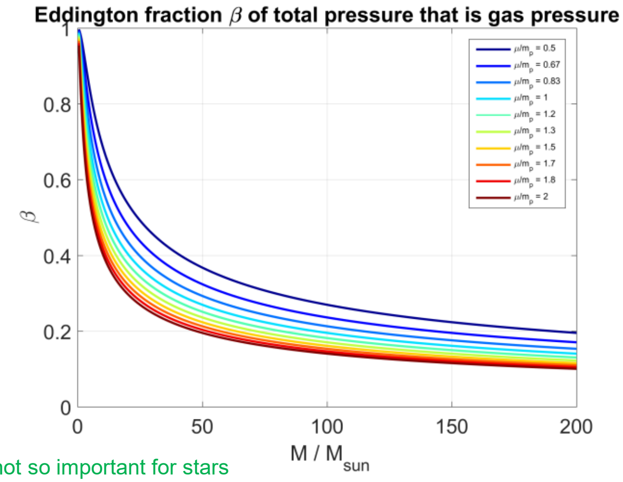
$$\phi = \tfrac{12}{\pi}\frac{\Lambda^3 k_B^4 c}{\sigma G^3\mu^4 M_\odot^2}$$

$$\therefore \left(\frac{M}{M_\odot}\right)^2\beta^4 + \phi\beta - \phi = 0$$

$$\frac{\mu}{m_p} = \frac{4}{6X+Y+2}$$



Solve using a numeric root-finding method



Eddington limit for star mass: When radiation pressure balances gravitational attraction

$$\frac{dT}{dr} \approx -\frac{3\rho\kappa l}{64\pi r^2 \sigma T^3}$$

Radiative energy transport. For hot stars, assume constant opacity $\kappa \approx 0.02(1+X)\ \mathrm{m^2kg^{-1}}$ (Hydrogen mass fraction: X)

$$P = \frac{4\sigma}{3c}T^4$$

If radiation pressure dominates

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$

Hydrostatic equilibrium

$$\therefore \frac{dP}{dr} = \frac{dP}{dT} \times \frac{dT}{dr}$$

$$\frac{dP}{dT} = \frac{16\sigma}{3c}T^3$$

$$\therefore \frac{dP}{dr} = \frac{16\sigma}{3c}T^3 \times \frac{dT}{dr} = -\frac{16\sigma}{3c}T^3 \frac{3\rho\kappa l}{64\pi r^2 \sigma T^3}$$

$$\therefore \frac{dP}{dr} = -\frac{\rho\kappa l}{4\pi r^2 c}$$

$$\therefore \frac{\rho\kappa l}{4\pi r^2 c} = \frac{Gm\rho}{r^2}$$

$$\therefore l = \frac{Gm\rho 4\pi r^2 c}{r^2 \rho\kappa} = \frac{4\pi c G m}{\kappa}$$

$$\therefore L_{\max} = \frac{4\pi c G M_{\max}}{\kappa}$$

Arthur Eddington (1882-1944)

From homology, for the most luminous stars

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^3$$

Consistent!

Line of best fit yields

$$\frac{L}{L_\odot} \approx 5.78\left(\frac{M}{M_\odot}\right)^3$$

$$\therefore L_{\max} = L_\odot \left(\frac{M_{\max}}{M_\odot}\right)^3$$

$$\therefore \frac{4\pi c G M_{\max}}{\kappa} = L_\odot \left(\frac{M_{\max}}{M_\odot}\right)^3$$

$$\therefore M_{\max} = \sqrt{\frac{4\pi c G M_\odot^3}{\kappa L_\odot}} = M_\odot \sqrt{\frac{4\pi c G M_\odot}{\kappa L_\odot}}$$

$$\therefore M_{\max} = M_\odot \sqrt{\frac{4\pi \times 3.00\times10^8 \times 6.67\times10^{-11} \times 1.99\times10^{30}}{0.02\times(1+0.7)\times 3.846\times10^{26}}}$$

$$\therefore M_{\max} \approx 190 M_\odot$$

guess that H abundance is 70% i.e. similar to the Sun

$$M_\odot = 1.989\times10^{30}\ \mathrm{kg}$$
$$R_\odot = 696{,}340\ \mathrm{km}$$
$$L_\odot = 3.846\times10^{26}\ \mathrm{Js^{-1}}$$
$$T_\odot = 5778\ \mathrm{K}$$

This is actually a bit of an overestimate. **A better estimate is 100 to 120 solar masses.** For the hottest stars, electron scattering is not the only source of opacity. The effects of incomplete ionization in the atmospheres of the hottest stars increases the opacity, which in the simple model of constant opacity used above, means the maximum star mass should *decrease*. We would also need to model the effect of varying Y and Z values too on the opacity model, since at high temperature, scattering resulting from bound-bound transitions in non-hydrogen atoms might contribute to higher opacity. (Pettini Lecture 10).

$$\frac{L}{L_\odot} \approx \begin{cases} 0.23\left(M/M_\odot\right)^{2.3} & M/M_\odot < 0.43 \\ \left(M/M_\odot\right)^4 & 0.43 < M/M_\odot < 2 \\ 1.4\left(M/M_\odot\right)^{3.5} & 2 < M/M_\odot < 55 \\ 32{,}000\, M/M_\odot & M/M_\odot > 55 \end{cases}$$

Empirical luminosity vs mass relationships from binary star observations

Lower limit for (main sequence) star mass

Sun core temperature $T_0 = 1.5\times10^7\ \mathrm{K}$

Minimum temperature for nuclear reactions (pp chain) to occur

$$T_{\min} \approx 4\times10^6\ \mathrm{K}$$

From homology argument, core temperature scales with mass (for low temperature stars undergoing pp chain fusion)

$$T_0 \propto M^{0.92}$$

Hence:

$$\frac{T_{\min}}{T_0} \approx \left(\frac{M_{\min}}{M_\odot}\right)^{0.92} \Rightarrow M_{\min} \approx \left(\tfrac{4}{15}\right)^{\frac{1}{0.92}} M_\odot \approx 0.2 M_\odot$$

Note if use constant opacity: $T_0 \propto M^{0.57}$

$$\therefore M_{\min} \approx \left(\tfrac{4}{15}\right)^{\frac{1}{0.57}} M_\odot \approx 0.1 M_\odot$$

According to Pettini's notes (Lecture10), the limit is more like 0.08 solar masses. **Wolf359** is a **red dwarf** in the constellation of Leo and is calculated to have a mass of 0.09 solar masses.

$G = 6.674\times10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$ $\quad R = 8.314\ \mathrm{Jmol^{-1}K^{-1}}$ $\quad \sigma = 5.67\times10^{-8}\ \mathrm{Wm^{-2}K^{-4}}$

The gravitational potential energy (GPE) of a polytropic star

Firstly calculate the GPE of a spherical mass of ***constant density*** (this is a polytropic index n of zero)

$$dU = -\frac{Gm}{r}\times 4\pi r^2 \rho\, dr \quad \therefore U = -4\pi G\int_0^R m r \rho\, dr$$

$$\rho = \text{constant}$$

$$\Rightarrow \rho = \frac{M}{\frac{4}{3}\pi R^3},; \quad m = \tfrac{4}{3}\pi r^3 \rho$$

$$\therefore U = -4\pi G\int_0^R \left(\tfrac{4}{3}\pi r^3 \rho\right) r \rho\, dr$$

$$\therefore U = -4\pi G\left(\frac{M}{\frac{4}{3}\pi R^3}\right)^2 \tfrac{4}{3}\pi \int_0^R r^4 dr$$

$$\therefore U = -4\pi G\left(\frac{M}{\frac{4}{3}\pi R^3}\right)^2 \tfrac{4}{3}\pi \tfrac{1}{5} R^5$$

$$\therefore U = -\tfrac{3}{5}\frac{GM^2}{R}$$

$n = 0$

It can be shown that this is a *special case* of the more general **Betti-Ritter formula** for a polytropic star of polytropic index n

$$U = -\frac{3}{5-n}\frac{GM^2}{R}$$

Proof of Betti-Ritter formula

$$U = -\int_0^R \frac{Gm\,dm}{r} = -\tfrac{1}{2} G\int_0^R \frac{dm^2}{r}$$

Build up GPE by assembling shells of mass dm

$$d\left(\frac{m^2}{r}\right) = \frac{r\,dm^2 - m^2 dr}{r^2} \Rightarrow \frac{dm^2}{r} = d\left(\frac{m^2}{r}\right) + \frac{m^2}{r^2}dr$$

$$\therefore U = -\tfrac{1}{2} G\int_0^R \left(d\left(\frac{m^2}{r}\right) + \frac{m^2}{r^2}dr \right) = -\tfrac{1}{2}\frac{GM^2}{R} - \tfrac{1}{2}\int_0^R \frac{Gm^2}{r^2}dr$$

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$

Hydrostatic equilibrium

$$\therefore -\frac{Gm}{r^2} = \frac{dP}{\rho}$$

$$\therefore U = -\tfrac{1}{2}\frac{GM^2}{R} + \tfrac{1}{2}\int_0^R \frac{m}{\rho}dP$$

Consider polytropic pressure vs density variation:

$$P = a\rho^{1+\frac{1}{n}} \quad \therefore \frac{dP}{\rho} = \left(\frac{n+1}{n}\right) a \rho^{\frac{1}{n}-1} d\rho$$

$$\frac{P}{\rho} = a\rho^{\frac{1}{n}} \quad \therefore d\left(\frac{P}{\rho}\right) = \tfrac{1}{n} a\rho^{\frac{1}{n}-1} d\rho$$

$$\therefore \frac{dP}{\rho} = (n+1)\, d\left(\frac{P}{\rho}\right)$$

Hence: $$U = -\tfrac{1}{2}\frac{GM^2}{R} + \tfrac{1}{2}(n+1)\int_0^R m\, d\left(\frac{P}{\rho}\right)$$

Integrating by parts:

$$\int_0^R m\, d\left(\frac{P}{\rho}\right) = \left[m\frac{P}{\rho}\right]_0^R - \int_0^R \frac{P}{\rho}dm$$

Assume pressure tends to zero at the surface of the star, and the mass enclosed is definitely zero at the centre of the star.

Hence: $$\left[m\frac{P}{\rho}\right]_0^R = 0$$

$$\therefore \int_0^R m\, d\left(\frac{P}{\rho}\right) = -\int_0^R \frac{P}{\rho}dm$$

Shell volume element $dV = \dfrac{dm}{\rho}$

$$\therefore \int_0^R m\, d\left(\frac{P}{\rho}\right) = -\int_0^R \frac{P}{\rho}dm = -\int_0^R P\,dV$$

Now $$d(PV) = P\,dV + V\,dP$$

$$\therefore \int_0^R d(PV) = \int_0^R P\,dV + \int_0^R V\,dP$$

If pressure is zero at star surface and enclosed volume is zero at the centre of the star:

$$\int_0^R d(PV) = 0$$

So $$\int_0^R d(PV) = 0 \Rightarrow -\int_0^R P\,dV = \int_0^R V\,dP$$

Hence: $$U = -\tfrac{1}{2}\frac{GM^2}{R} + \tfrac{1}{2}(n+1)\int_0^R m\, d\left(\frac{P}{\rho}\right)$$

$$U = -\tfrac{1}{2}\frac{GM^2}{R} - \tfrac{1}{2}(n+1)\int_0^R P\,dV$$

$$U = -\tfrac{1}{2}\frac{GM^2}{R} + \tfrac{1}{2}(n+1)\int_0^R V\,dP$$

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$

$$U = -\tfrac{1}{2}\frac{GM^2}{R} + \tfrac{1}{2}(n+1)\int_0^R \tfrac{4}{3}\pi r^3 \left(\frac{-Gm\rho}{r^2}\right)dr$$

$$V = \tfrac{4}{3}\pi r^3$$

$$U = -\tfrac{1}{2}\frac{GM^2}{R} + \tfrac{1}{6}(n+1)\int_0^R -\frac{Gm}{r}\times 4\pi r^2 \rho\, dr$$

$$dm = 4\pi r^2 \rho$$

$$U = -\tfrac{1}{2}\frac{GM^2}{R} - \tfrac{1}{6}(n+1)\int_0^R \frac{Gm\,dm}{r}$$

$$U = -\int_0^R \frac{Gm\,dm}{r}$$

$$U = -\tfrac{1}{2}\frac{GM^2}{R} + \tfrac{1}{6}(n+1)U$$

$$U\left(1 - \tfrac{1}{6}n - \tfrac{1}{6}\right) = -\tfrac{1}{2}\frac{GM^2}{R}$$

$$U(6-n-1) = -3\frac{GM^2}{R}$$

$$\therefore U = -\frac{3}{5-n}\frac{GM^2}{R}$$

Note we could use this to find the total energy

Virial theorem states that force between any two particles results from a potential of the form

$$V = \frac{k}{r^{\alpha}}$$ then $$\langle \text{KE}\rangle = -\tfrac{\alpha}{2}\langle \text{PE}\rangle$$

E is total energy

$$E = \langle \text{KE}\rangle + \langle \text{PE}\rangle \quad \therefore E = \left(1-\tfrac{\alpha}{2}\right)\langle \text{PE}\rangle$$

Note we have *not needed* to invoke any ideal gas assumptions! For an isothermal change involving an ideal gas:

$$PV = Nk_BT \Rightarrow d(PV) = 0 \quad \text{if } dT = 0 \qquad \int_0^R d(PV) = 0$$

Integrand zero throughout in this case, not just integral at limits

Degeneracy pressure and the Chandrasekhar limit for the mass of white dwarf stars

A Main Sequence star of mass less than about 8 solar masses will eventually swell to a **red giant** and eventually dissipate, leaving a **white dwarf** and ultimately a **black dwarf**. *Unless* the red giant forms a **binary** with a white dwarf and transfers mass such that the **white dwarf** exceed 1.44 solar masses. This results in a **Type 1a supernova**, with no remnant.

Let's firstly explore this limit using the **Lane-Emden equation**. It can be shown that the more massive white dwarf stars can be modelled by polytropes of index $n=3$ (i.e. just like MS stars). Lane-Emden suggests a *condensation* of 53.95 for these types of stars.

n	0	1	2	3	4	5
ψ_0	$\sqrt{6}$	π	4.35	6.89	14.93	∞
$-\psi_0^2 \left.\frac{d\theta}{d\psi}\right\vert_{\psi_0}$	$2\sqrt{6}$	π	2.41	2.02	1.80	1.7
$\frac{\rho_0}{\bar{\rho}}$	1	$\frac{1}{3}\pi^2$	11.37	53.95	617.50	∞

$$R=\alpha\psi_0$$
$$M=4\pi\alpha^3\rho_0\Lambda$$
$$\Lambda=-\psi_0^2\left.\frac{d\theta}{d\psi}\right|_{\psi_0}$$

Let's assume a maximum core density such that the 'nuclei are (almost) touching.' This means a minimum spacing of about $x = 10^{-15}$ m, but it could be rather more than this for the potential for runaway carbon fusion.

$$\rho_0=\frac{m_p}{\frac{4}{3}\pi x^3}<\frac{1.67\times10^{-27}}{\frac{4}{3}\pi\left(10^{-15}\right)^3}\ \text{kgm}^{-3}\approx 4.0\times10^{17}\ \text{kgm}^{-3}$$

Assuming white dwarfs are mostly carbon and oxygen $X,Y=0$ meaning average molecular mass per electron is: $\mu=\frac{4m_p}{6X+Y+2}\approx 2m_p$

Lane-Emden predicts the following formula for the star mass $M=4\pi\alpha^3\rho_0\Lambda$

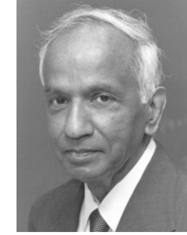
$$\Lambda=-\psi_0^2\left.\frac{d\theta}{d\psi}\right|_{\psi_0} \qquad \alpha=\sqrt{\frac{k_BT_0(3+1)}{4\pi G\mu\rho_0}}$$

$$\therefore M=4\pi\left(\frac{k_BT_0(3+1)}{4\pi G\mu\rho_0}\right)^{\frac{3}{2}}\rho_0\Lambda$$

$$\therefore \frac{M^2}{16\pi^2\Lambda^2}=\left(\frac{k_BT_0}{\pi G\mu}\right)^3\frac{1}{\rho_0}$$

$$\therefore \rho_0=\frac{16\pi^2\Lambda^2}{M^2}\left(\frac{k_BT_0}{\pi G\mu}\right)^3$$

But this assumes an **ideal gas at the core** … This is perhaps **not** a good model for white dwarfs near the Chandrasekhar limit, as **electron degeneracy pressure** will become the dominant factor which resists gravitational collapse.



Subrahmanyan Chandrasekhar (1910-1995)

$$\rho_0=\frac{12m_p}{\frac{4}{3}\pi x^3}$$

Assuming carbon atoms are in a white dwarf, what is their spacing x in the core?

$$\therefore \frac{12m_p}{\frac{4}{3}\pi x^3}=\frac{16\pi^2\Lambda^2}{M^2}\left(\frac{k_BT_0}{\pi G\mu}\right)^3$$

$$\therefore x=\sqrt[3]{\frac{9m_pM^2}{16\pi^3\Lambda^2}\left(\frac{\pi G\mu}{k_BT_0}\right)^3}$$

Let's assume a white dwarf explodes when the core temperature is high enough for carbon fusion, i.e. $T_0\approx 6\times10^8$ K

Using $M=1.4M_\odot$

$\therefore x=3.3\times10^{-12}\text{m}$ ← This is two orders of magnitude smaller than atomic size, but about 300 proton radii

$\therefore R=0.09R_\odot$ ← This is about 10x larger than the 'official' values

$\therefore \rho_0=1.4\times10^8\ \text{kgm}^{-3}$ ← This is about 10x smaller than the 'official' values

Note: $R\propto\alpha\propto\sqrt{\frac{T_0}{\rho_0}}$

$\rho_0\propto T_0^3 \Rightarrow R\propto\sqrt{\frac{T_0}{T_0^3}}$

$\therefore R\propto\frac{1}{T_0}$ For a fixed star mass, given Lane-Emden model and Assuming an ideal gas.

i.e. let's try a slightly higher temperature!

$T_0=9\times6\times10^8$ K

$\therefore x=3.7\times10^{-13}\text{m}$

$\therefore R=0.01R_\odot$

$\therefore \rho_0=9.4\times10^{10}\ \text{kgm}^{-3}$

So we can get sensible numbers for carbon nuclei spacing, white dwarf radius and density (see more accurate calculations on the next few pages)

Derivation of the Chandrasekhar limit will firstly require a model of **degeneracy pressure**

Although a white dwarf is no longer hot enough for fusion to occur, assume its constituent (probably carbon and oxygen) atoms are hot enough for electrons to be ionized. We shall model a white dwarf as being a sphere of mass M and radius R with a 'sea' of free electrons occupying the space between the nuclei. Consider a cube of side length L of this 'electron sea.' Let's assume the wavefunction of electrons has a de-Broglie wavelength such that whole number multiples equal L

$$\therefore L=n_x\lambda_x,\quad n_x=1,2,3...$$

Using the de-Broglie relationship: $p_x=h/\lambda_x$ hence $p_x=\frac{1}{L}n_xh$

Planck's constant $h=6.626\times10^{-34}$ Js

The **number of momentum states** between p_x and p_x+dp_x is $dn_x=\frac{Ldp_x}{h}$

Generalizing to 3D: $dn_xdn_ydn_z=\frac{L^3dp_xdp_ydp_z}{h^3}=\frac{L^3}{h^3}4\pi p^2dp$

i.e. a shell in momentum space

$p=\sqrt{p_x^2+p_y^2+p_z^2}$

Now electrons are **fermions**, which means a maximum of two electrons can share the same momentum state in one region of space, as long as their spins are opposite. This is the **Pauli Exclusion Principle**.

Wolfgang Pauli (1900-1958)

Hence the (maximum) number density of momentum states is:

$$\frac{dn_x dn_y dn_z}{L^3} = \frac{2\times 4\pi p^2 dp}{h^3} = n(p)dp$$

$$\therefore n(p)dp = \frac{8\pi p^2}{h^3}dp$$

Let's assume the maximum momentum in the electron system is p_F (we'll call this the **Fermi momentum**), this means the total electron density is

$$n_e = \int_0^{p_F} n(p)dp = \frac{8\pi}{h^3}\int_0^{p_F} p^2 dp = \frac{8\pi}{3h^3}p_F^3$$

$$\therefore p_F = \tfrac{1}{2}\left(\tfrac{3}{\pi}\right)^{1/3} n_e^{1/3} h$$

Enrico Fermi (1901-1954)

If the electrons are moving at speed v, we can calculate the pressure P resulting from the *rate of change of momentum of electrons*. i.e. consider a tube of 1m² cross section of length v, with momentum p, and $vn(p)dp$ electrons with this momentum

$$\therefore P = \int_0^{p_F} \tfrac{1}{3} pvn(p)dp$$

Note the factor of 1/3 to average over x,y,z directions, since, by Cartesian to spherical polar conversion

$$\tfrac{1}{3}\int (p_x v_x + p_y v_y + p_z v_z) dp_x dp_y dp_z = \tfrac{1}{3}\int pv\times 4\pi p^2 dp \qquad n(p)dp = \frac{8\pi p^2}{h^3}dp$$

In the non-relativistic limit: $v = p/m_e$

$$\therefore P = \int_0^{p_F}\tfrac{1}{3}pvn(p)dp = \int_0^{p_F}\tfrac{1}{3}\frac{p^2}{m_e}\frac{8\pi p^2}{h^3}dp$$

$$\therefore P = \frac{8\pi}{3m_e h^3}\int_0^{p_F} p^4 dp = \frac{8\pi}{15 m_e h^3}p_F^5$$

$$\therefore P = \frac{8\pi}{15 m_e h^3}\left(\tfrac{1}{2}\left(\tfrac{3}{\pi}\right)^{1/3} n_e^{1/3} h\right)^5 \qquad p_F = \tfrac{1}{2}\left(\tfrac{3}{\pi}\right)^{1/3} n_e^{1/3} h$$

$$\therefore P = \frac{8\pi}{480}\left(\tfrac{3}{\pi}\right)^{\frac{5}{3}}\frac{h^2}{m_e}n_e^{5/3} = \tfrac{1}{20}\tfrac{\pi}{3}\left(\tfrac{3}{\pi}\right)^{\frac{5}{3}}\frac{h^2}{m_e}n_e^{5/3}$$

$$\therefore P = \tfrac{1}{20}\left(\tfrac{3}{\pi}\right)^{\frac{2}{3}}\frac{h^2}{m_e}n_e^{5/3}$$

As a white dwarf increases in mass, we might expect it to get hotter and therefore the electrons to move faster. Eventually they will reach *relativistic speeds*, which has a **limit of the speed of light.**

In this limit:

$$P \to \int_0^{p_F}\tfrac{1}{3}pcn(p)dp = \int_0^{p_F}\tfrac{1}{3}pc\frac{8\pi p^2}{h^3}dp$$

$$= \frac{8\pi c}{3h^3}\int_0^{p_F}p^3 dp = \frac{8\pi c}{12h^3}p_F^4 = \frac{2\pi c}{3h^3}p_F^4$$

$$= \frac{2\pi c}{3h^3}\left(\tfrac{1}{2}\left(\tfrac{3}{\pi}\right)^{1/3}n_e^{1/3}h\right)^4$$

$$\therefore P = \tfrac{1}{8}\left(\tfrac{3}{\pi}\right)^{1/3} hcn_e^{4/3}$$

$M_\odot = 1.988\times 10^{30}$ kg
$k_B = 1.381\times 10^{-23}$ JK^{-1}
$G = 6.674\times 10^{-11}$ m^{3}kg^{-1}s^{-2}
$\sigma = 5.670\times 10^{-8}$ Wm^{-2}K^{-4}
$m_p = 1.673\times 10^{-27}$ kg
$m_e = 9.109\times 10^{-31}$ kg
$c = 2.998\times 10^{8}$ ms^{-1}
$h = 6.626\times 10^{-34}$ Js

Recall the **Lane-Emden** results for a polytropic star

$$P = a\rho^{1+\frac{1}{n}}$$

n	1.5	3
ψ_0	3.65	6.90
$-\psi_0^2 \left.\frac{d\theta}{d\psi}\right\vert_{\psi_0}$	2.71	2.02
$\frac{\rho_0}{\bar{\rho}}$	5.99	54.16

$$R = \alpha\psi_0$$
$$M = 4\pi\alpha^3\rho_0\Lambda$$
$$\Lambda = -\psi_0^2\left.\frac{d\theta}{d\psi}\right|_{\psi_0}$$
$$\frac{\rho_0}{\bar{\rho}} = \frac{\psi_0^3}{3\Lambda}$$
$$\alpha = \sqrt{\frac{a(n+1)}{4\pi G}\rho_0^{\frac{1}{n}-1}}$$

If the mass per electron ionized is μ then white dwarf density $\rho = n_e\mu$

for a mostly C,O composition $\mu \approx 2m_p$

We can therefore express this **electron degeneracy pressure** in terms of white dwarf density:

$$n_e = \rho/\mu$$

$$\therefore P = \begin{cases} \tfrac{1}{20}\left(\tfrac{3}{\pi}\right)^{\frac{2}{3}}\dfrac{h^2}{m_e\mu^{5/3}}\rho^{1+\frac{1}{3/2}} & \text{classical} \\ \tfrac{1}{8}\left(\tfrac{3}{\pi}\right)^{1/3}\dfrac{hc}{\mu^{4/3}}\rho^{1+\frac{1}{3}} & \text{relativistic}\end{cases}$$

i.e. polytrope of $n = 3/2$

i.e. polytrope of $n = 3$

$$R^2 = \alpha^2\psi_0^2$$

$$R^2 = \frac{a(n+1)}{4\pi G}\rho_0^{\frac{1}{n}-1}\psi_0^2$$

$$R^2 = \frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\frac{M}{\frac{4}{3}\pi R^3}\right)^{\frac{1}{n}-1}\psi_0^2$$

$$R^{2+\frac{3}{n}-3} = \frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{n}}\psi_0^2 M^{\frac{1}{n}-1}$$

$$R = \left(\frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{n}}\psi_0^2\right)^{\frac{1}{\frac{3}{n}-1}} M^{\frac{\frac{1}{n}-1}{\frac{3}{n}-1}}$$

$$\therefore R = \left(\frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{n}}\psi_0^2\right)^{\frac{n}{3-n}} M^{\frac{n-1}{n-3}}$$

Radius vs mass relationship for a polytropic star

This is *undefined* for $n = 3$ (see next page!)

For **classical limit**, using $n = 3/2$ $\quad a = \tfrac{1}{20}\left(\tfrac{3}{\pi}\right)^{\frac{2}{3}}\dfrac{h^2}{m_e\mu^{5/3}}$

$$R = \left(\frac{9}{8192\pi^4}\right)^{\frac{1}{3}}\psi_0\Lambda^{\frac{1}{3}}\frac{h^2}{Gm_e\mu^{\frac{5}{3}}} \approx 7762\left(\frac{M}{1.44M_\odot}\right)^{-1/3}\text{ km}$$

So a white dwarf approaching the Chandrasekhar limit is about the size of the Earth (6371 km).

A strange result! The more massive a white dwarf is, the smaller it gets!

For the **relativistic case** (n=3)

$$R^{2+\frac{3}{n}-3} = \frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{n}}\psi_0^2 M^{\frac{1}{n}-1}$$

$$n=3 \Rightarrow 2+\tfrac{3}{n}-3=0$$

$$1 = \frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{n}}\psi_0^2 M^{\frac{1-n}{n}}$$ only if $n=3$

$$\therefore M = \left(\frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{n}}\psi_0^2\right)^{\frac{n}{n-1}}$$

$$\therefore M = \frac{\Lambda\sqrt{3}}{\pi\sqrt{32}}\left(\frac{hc}{G\mu^{\frac{4}{3}}}\right)^{\frac{3}{2}} \approx 1.44 M_{\odot}$$

Which is the **Chandrasekhar Limit.** If a white dwarf's mass exceeds this, then runaway carbon fusion will result in a Type1A supernova, with no remnant!

$$a = \tfrac{1}{8}\left(\tfrac{3}{\pi}\right)^{1/3}\frac{hc}{\mu^{4/3}}$$

$$M = \left(\frac{a(n+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{n}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{n}}\psi_0^2\right)^{\frac{n}{n-1}}$$

$$M = \left(\frac{\tfrac{1}{8}\left(\tfrac{3}{\pi}\right)^{1/3}\frac{hc}{\mu^{4/3}}(3+1)}{4\pi G}\left(\frac{\psi_0^3}{3\Lambda}\right)^{\frac{1}{3}-1}\left(\tfrac{4}{3}\pi\right)^{1-\frac{1}{3}}\psi_0^2\right)^{\frac{3}{3-1}}$$

$$M = \left(\frac{hc}{G\mu^{\frac{4}{3}}}\right)^{\frac{3}{2}}\left(\psi_0^{-3\times\frac{2}{3}+2}\right)^{3/2}\Lambda^{\frac{2}{3}\frac{3}{2}}\left[2^{-1-2+4/3}\times 3^{\frac{1}{3}-\frac{2}{3}\frac{2}{3}}\pi^{-\frac{1}{3}-1+\frac{2}{3}}\right]^{\frac{2}{3}}$$

$$M = \left(\frac{hc}{G\mu^{\frac{4}{3}}}\right)^{\frac{3}{2}}\frac{\Lambda}{\pi}^{\frac{2}{3}}\left(\sqrt{3}\times\left(2^{-\frac{5}{3}}\right)^{\frac{3}{2}}\right)$$

$$\therefore M = \frac{\Lambda\sqrt{3}}{\pi\sqrt{32}}\left(\frac{hc}{G\mu^{\frac{4}{3}}}\right)^{\frac{3}{2}}$$

n	1.5	3
ψ_0	3.65	6.90
Λ	2.71	2.02
$\frac{\rho_0}{\bar{\rho}}$	5.99	54.16

Lane-Emden results

Returning to the classical limit:

$$R \approx 7762\left(\frac{M}{1.44M_{\odot}}\right)^{-1/3}\text{km}$$

$$\therefore \frac{M}{1.44M_{\odot}} = \left(\frac{R}{7762\,\text{km}}\right)^{-3}$$

$$\therefore \frac{M}{1.44M_{\odot}}\tfrac{4}{3}\pi R^3 = \left(\frac{R}{7762\,\text{km}}\right)^{-3}\tfrac{4}{3}\pi R^3$$

$$\therefore M\tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi \times 1.44M_{\odot}\times(7762\,\text{km})^3$$

$$\therefore \frac{M}{M_{\odot}}\frac{\tfrac{4}{3}\pi R^3}{\tfrac{4}{3}\pi R_{\oplus}^3} = 1.44\times\left(\frac{7762}{6371}\right)^3 \approx 2.60$$

So the product of a white dwarf volume and mass is a constant.

$M_{\odot} = 1.988\times10^{30}$ kg
$k_B = 1.381\times10^{-23}$ JK^{-1}
$G = 6.674\times10^{-11}$ m^{3}kg^{-1}s^{-2}
$\sigma = 5.670\times10^{-8}$ Wm^{-2}K^{-4}
$m_p = 1.673\times10^{-27}$ kg
$m_e = 9.109\times10^{-31}$ kg
$c = 2.998\times10^{8}$ ms^{-1}
$h = 6.626\times10^{-34}$ Js

```
*****************************************************************************
Chandrasekhar mass = 1.4366 solar masses.
 White dwarf radius R (in km) = ( 7762.3281 km )* (M/M_ch)^(-1/3)
 White dwarf density at core assuming classical n=3/2 model: 8.73e+09 kg/m^3
 White dwarf density at core assuming relativistic n=3 model: 7.9e+10 kg/m^3
 White dwarf mass in solar masses x white dwarf volume in Earth volumes 2.598
*****************************************************************************
```

MATLAB code outputs

$R_{\odot} = 696340\,\text{km} \quad \therefore 7762\,\text{km} \approx 0.01R_{\odot}$

Looking at the Chandrasekhar limit from an energy perspective

To keep thing simple, we'll assume a *constant density*, rather than use the Lane-Emden results.

$$P = \begin{cases} \tfrac{1}{20}\left(\tfrac{3}{\pi}\right)^{\frac{2}{3}}\frac{h^2}{m_e\mu^{5/3}}\rho^{\frac{5}{3}} & \text{classical} \\ \tfrac{1}{8}\left(\tfrac{3}{\pi}\right)^{1/3}\frac{hc}{\mu^{4/3}}\rho^{\frac{4}{3}} & \text{relativistic} \end{cases}$$

(assuming KE contribution is degeneracy (1/3) pressure x volume)

Perhaps a better approximation is to use $\tfrac{3}{2}PV$ than just PV as in Pettini.

Betti-Ritter formula

$$E = P\times\tfrac{4}{3}\pi R^3 - \tfrac{3}{5-n}\frac{GM^2}{R}$$

$$\therefore E = \begin{cases} \tfrac{3}{2}\tfrac{1}{20}\left(\tfrac{3}{\pi}\right)^{\frac{2}{3}}\frac{h^2}{m_e\mu^{5/3}}\left(\frac{M}{\tfrac{4}{3}\pi R^3}\right)^{\frac{5}{3}}\tfrac{4}{3}\pi R^3 - \tfrac{6}{7}\frac{GM^2}{R} & \text{classical} \\ \tfrac{3}{2}\tfrac{1}{8}\left(\tfrac{3}{\pi}\right)^{1/3}\frac{hc}{\mu^{4/3}}\left(\frac{M}{\tfrac{4}{3}\pi R^3}\right)^{\frac{4}{3}}\tfrac{4}{3}\pi R^3 - \tfrac{3}{2}\frac{GM^2}{R} & \text{relativistic} \end{cases}$$

$$\therefore E = \begin{cases} \frac{A_c}{R^2} - \frac{B_c}{R} & \text{classical} \\ \frac{A_r - B_r}{R} & \text{relativistic} \end{cases}$$

Non-relativistic limit has an **energy minima** at:

$$\frac{dE}{dr} = 0 \Rightarrow -\frac{2A_c}{R^3} + \frac{B_c}{R^2} = 0 \Rightarrow \frac{2A_c}{R^3} = \frac{B_c}{R^2} \Rightarrow R = \frac{2A_c}{B_c}$$

$$A_c = \tfrac{3}{2}\tfrac{4}{3}\pi\tfrac{1}{20}\left(\tfrac{3}{\pi}\right)^{\frac{2}{3}}\frac{h^2}{m_e\mu^{5/3}}\left(\frac{M}{\tfrac{4}{3}\pi}\right)^{\frac{5}{3}}; \quad B_c = \tfrac{6}{7}GM^2$$

$$A_r = \tfrac{3}{2}\tfrac{1}{8}\tfrac{4}{3}\pi\left(\tfrac{3}{\pi}\right)^{1/3}\frac{hc}{\mu^{4/3}}\left(\frac{M}{\tfrac{4}{3}\pi}\right)^{\frac{4}{3}}; \quad B_r = \tfrac{3}{2}GM^2$$

Relativistic energy expression has *no minima*, and negative when $B_r > A_r$. We can interpret this as the Chandrasekhar mass limit.

$$A_r = B_r \quad \therefore \tfrac{3}{2}\tfrac{1}{8}\tfrac{4}{3}\pi\left(\tfrac{3}{\pi}\right)^{1/3}\frac{hc}{\mu^{4/3}}\left(\frac{M}{\tfrac{4}{3}\pi}\right)^{\frac{4}{3}} = \tfrac{3}{2}GM^2$$

$$\therefore \tfrac{3}{2}\tfrac{1}{8}\tfrac{4}{3}\pi\left(\tfrac{3}{\pi}\right)^{1/3}\frac{hc}{\tfrac{3}{2}G\mu^{4/3}}\left(\frac{1}{\tfrac{4}{3}\pi}\right)^{\frac{4}{3}} = M^{\frac{2}{3}}$$

$$\therefore M = \left(\tfrac{3}{2}\tfrac{1}{8}\tfrac{4}{3}\pi\left(\tfrac{3}{\pi}\right)^{1/3}\frac{hc}{\tfrac{3}{2}G\mu^{4/3}}\left(\frac{1}{\tfrac{4}{3}\pi}\right)^{\frac{4}{3}}\right)^{\frac{3}{2}} = \frac{3}{\pi\sqrt{2048}}\left(\frac{hc}{G\mu^{\frac{4}{3}}}\right)^{\frac{3}{2}} \approx 0.154M_{\odot}$$

i.e. an underestimate from 1.44 solar masses.

Classical limit of total energy expression:

$$R = \frac{2A_c}{B_c}$$

$$A_c = \tfrac{3}{2}\tfrac{4}{3}\pi\tfrac{1}{20}\left(\tfrac{3}{\pi}\right)^{\frac{2}{3}}\frac{h^2}{m_e\mu^{5/3}}\left(\frac{M}{\frac{4}{3}\pi}\right)^{\frac{5}{3}}; \quad B_c = \tfrac{6}{7}GM^2$$

$$\therefore R = \frac{2\times\tfrac{3}{2}\tfrac{4}{3}\pi\tfrac{1}{20}\left(\tfrac{3}{\pi}\right)^{\frac{2}{3}}\frac{h^2}{m_e\mu^{5/3}}\left(\frac{M}{\frac{4}{3}\pi}\right)^{\frac{5}{3}}}{\tfrac{6}{7}GM^2}$$

$$\therefore R = \pi^{1-\frac{2}{3}-\frac{5}{3}}\times 2^{1+2-2-1-\frac{10}{3}-1}\times 3^{-1+\frac{2}{3}-1+\frac{5}{3}+1}\times 5^{-1}\times 7\frac{h^2}{Gm_e\mu^{5/3}}M^{-\frac{1}{3}}$$

$$\therefore R = \pi^{-\frac{4}{3}}\times 2^{-\frac{13}{3}}\times 3^{\frac{4}{3}}\times 5^{-1}\times 7\frac{h^2}{Gm_e\mu^{5/3}}M^{-\frac{1}{3}}$$

$$\therefore R = \left(\frac{343\times 81}{125\times 8192\pi^4}\right)^{\frac{1}{3}}\frac{h^2}{Gm_e\mu^{5/3}}M^{-\frac{1}{3}}$$

$$\therefore R \approx 4439\left(\frac{M}{1.44M_\odot}\right)^{-\frac{1}{3}}\text{ km}$$

This is a factor of 1.7 out compared to the Lane-Emden result

$$R = \left(\frac{9}{8192\pi^4}\right)^{\frac{1}{3}}\psi_0\Lambda^{\frac{1}{3}}\frac{h^2}{Gm_e\mu^{\frac{5}{3}}} \approx 7762\left(\frac{M}{1.44M_\odot}\right)^{-1/3}\text{ km}$$

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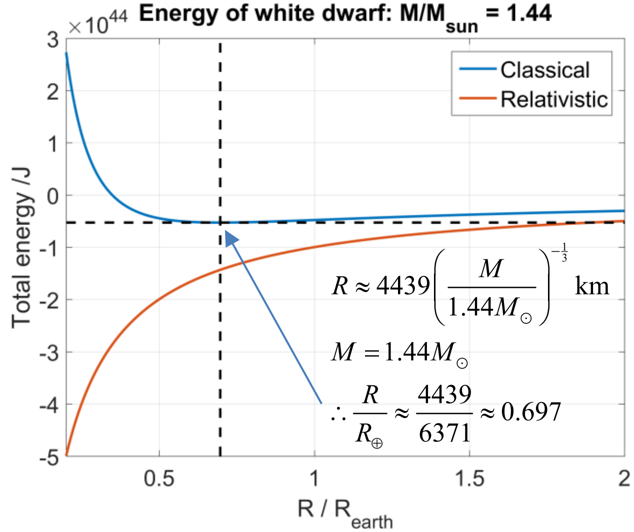
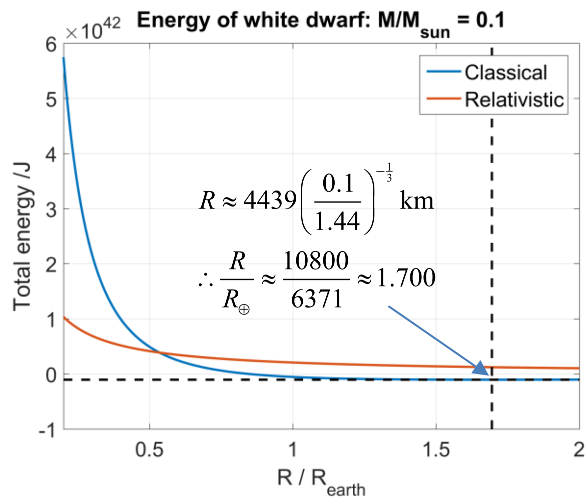
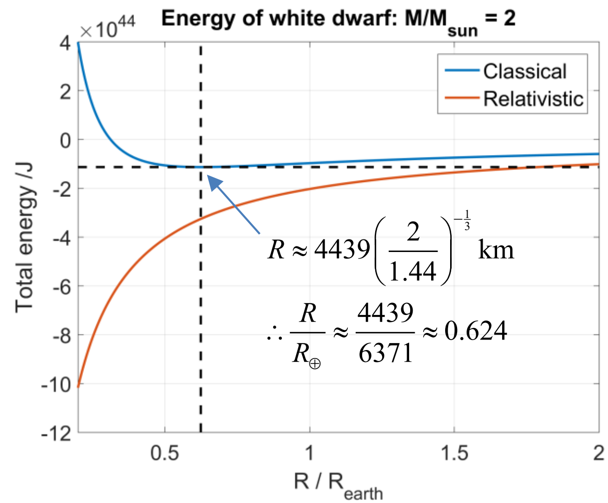


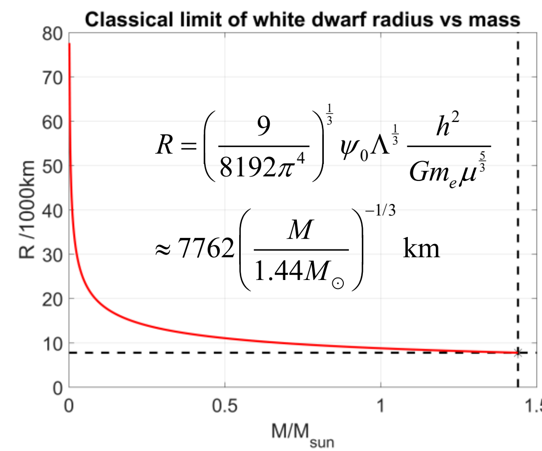


$$E = \begin{cases} \frac{A_c}{R^2} - \frac{B_c}{R} & \text{classical} \\ \frac{A_r - B_r}{R} & \text{relativistic} \end{cases}$$

$$A_c = \tfrac{3}{2}\tfrac{4}{3}\pi\tfrac{1}{20}\left(\tfrac{3}{\pi}\right)^{\frac{2}{3}}\frac{h^2}{m_e\mu^{5/3}}\left(\frac{M}{\frac{4}{3}\pi}\right)^{\frac{5}{3}}; \quad B_c = \tfrac{6}{7}GM^2$$

$$A_r = \tfrac{3}{2}\tfrac{1}{8}\tfrac{4}{3}\pi\left(\tfrac{3}{\pi}\right)^{1/3}\frac{hc}{\mu^{4/3}}\left(\frac{M}{\frac{4}{3}\pi}\right)^{\frac{4}{3}}; \quad B_r = \tfrac{3}{2}GM^2$$





For energy method Chandrasekhar mass estimate is:

$$\frac{3}{\pi\sqrt{2048}}\left(\frac{hc}{G\mu^{\frac{4}{3}}}\right)^{\frac{3}{2}} \approx 0.154M_\odot$$

which is 9.4 x smaller than 1.44 solar masses, which is the correct value.