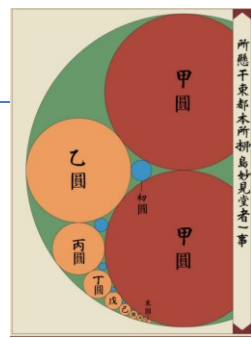
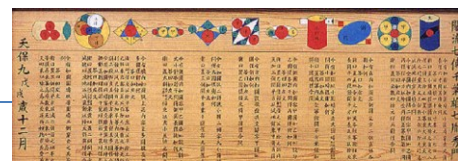
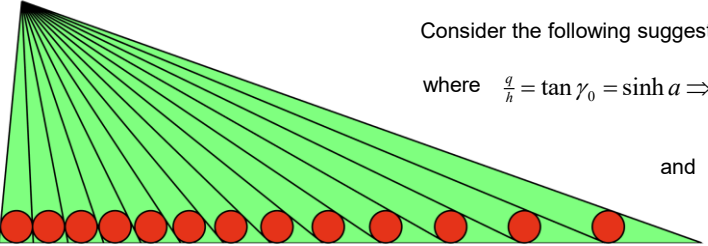


Sangaku or *san gaku* ('calculation tablet') are Japanese geometrical problems or theorems on wooden tablets which were placed as offerings at Shinto shrines or Buddhist temples during the Edo period by members of all social classes.



Sangaku #1: Equal incircles. Consider a triangle with vertices $(0,0)$, (p,h) , $(p+q,0)$. If rays are drawn from (p,h) to form a stack of triangles which form the same base (i.e. the horizontal axis), what should their bases be such that the incircles of each triangle has the same radius r ?



Consider the following suggested relation: $\tan \gamma_n = \sinh(a + nb)$

where $\frac{q}{h} = \tan \gamma_0 = \sinh a \Rightarrow a = \sinh^{-1}\left(\frac{q}{h}\right)$

and $b = 2 \tanh^{-1}\left(\frac{r}{h-r}\right)$

$$w = \frac{r}{\cos \phi} + \frac{r}{\cos \theta} = r(\sec \gamma_n + \sec \gamma_{n-1})$$

$$w = (h-r)(\tan \gamma_n - \tan \gamma_{n-1})$$

$$\sec^2 \gamma = 1 + \tan^2 \gamma = 1 + \sinh^2 \gamma = \cosh^2 \gamma$$

$$\therefore \sec \gamma = \cosh \gamma.$$

$$\therefore \sinh(a+nb) - \sinh(a+nb-b) = \frac{r}{h-r} \{ \cosh(a+nb) + \cosh(a+nb-b) \}$$

$$\sinh(a+nb) - \sinh(a+nb-b) = 2 \cosh\left(\frac{1}{2}(a+nb) + \frac{1}{2}(a+nb-b)\right) \sinh\left(\frac{1}{2}(a+nb) - \frac{1}{2}(a+nb-b)\right)$$

$$\sinh(a+nb) - \sinh(a+nb-b) = 2 \cosh\left(a+nb - \frac{1}{2}b\right) \sinh\left(\frac{1}{2}b\right)$$

$$\cosh(a+nb) + \cosh(a+nb-b) = 2 \cosh\left(\frac{1}{2}(a+nb) + \frac{1}{2}(a+nb-b)\right) \cosh\left(\frac{1}{2}(a+nb) - \frac{1}{2}(a+nb-b)\right)$$

$$\cosh(a+nb) + \cosh(a+nb-b) = 2 \cosh\left(a+nb - \frac{1}{2}b\right) \cosh\left(\frac{1}{2}b\right)$$

$$\therefore 2 \cosh\left(a+nb - \frac{1}{2}b\right) \sinh\left(\frac{1}{2}b\right) = \frac{r}{h-r} \times 2 \cosh\left(a+nb - \frac{1}{2}b\right) \cosh\left(\frac{1}{2}b\right)$$

$$\therefore \tanh\left(\frac{1}{2}b\right) = \frac{r}{h-r}$$

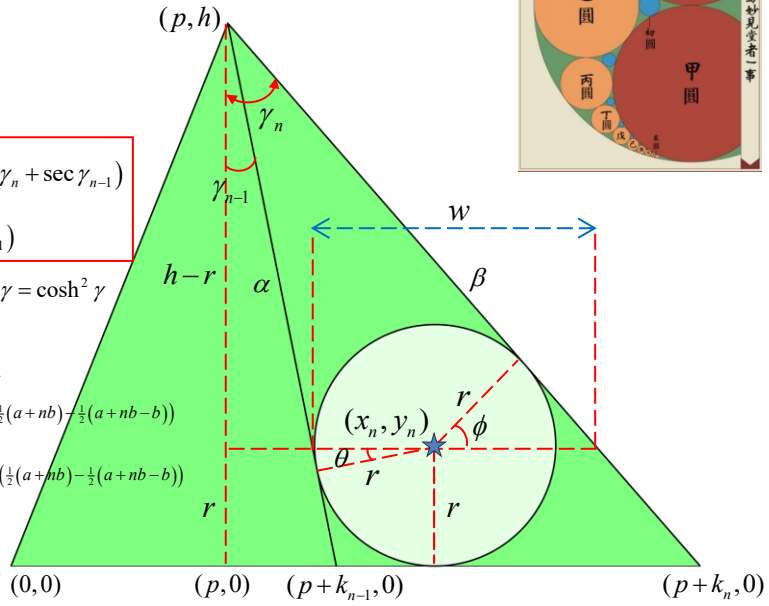
$$\therefore b = 2 \tanh^{-1}\left(\frac{r}{h-r}\right)$$

$$\sinh x + \sinh y = 2 \sinh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right)$$

$$\cosh x + \cosh y = 2 \cosh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right)$$

$$\sinh x - \sinh y = 2 \cosh\left(\frac{x+y}{2}\right) \sinh\left(\frac{x-y}{2}\right)$$

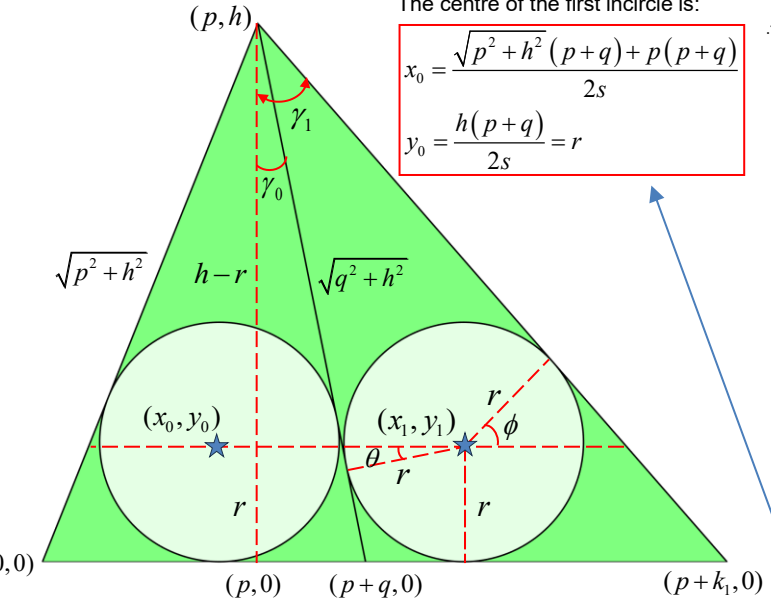
$$\cosh x - \cosh y = 2 \sinh\left(\frac{x+y}{2}\right) \sinh\left(\frac{x-y}{2}\right)$$



The centre of the first incircle is:

$$x_0 = \frac{\sqrt{p^2 + h^2}(p+q) + p(p+q)}{2s}$$

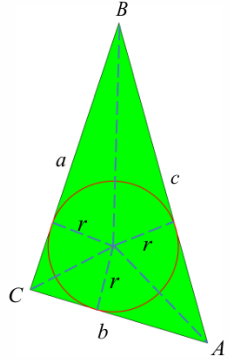
$$y_0 = \frac{h(p+q)}{2s} = r$$



From geometry: $\gamma_0 = \tan^{-1} \frac{q}{h}$, $k_1 = h \tan \gamma_1$, $\theta = \gamma_0$, $\phi = \gamma_1$.

$$\vec{OZ} = \frac{a\vec{OA} + b\vec{OB} + c\vec{OC}}{a+b+c}$$

Vector equation for **centre of inscribed circle** in terms of position vectors of triangle vertices.



Since the inscribed circle is tangent to the sides of the triangle, the triangle area is

$$\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}r(a+b+c)$$

Using Hero's formula:

$$rs = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

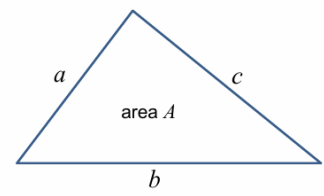
Radius of inscribed circle in terms of the length of the triangle edges.

From geometry: $k_n = h \tan \gamma_n$, $\theta = \gamma_{n-1}$, $\phi = \gamma_n$, $y_n = y_0 = r$.

and the horizontal coordinate of the centre of the n^{th} incircle is:

$$x_n = \frac{\beta(p+k_{n-1}) + \alpha(p+k_n)}{k_n - k_{n-1} + \alpha + \beta}$$

$$\alpha = \frac{h}{\cos \gamma_{n-1}}, \beta = \frac{h}{\cos \gamma_n}$$



$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{1}{2}(a+b+c)$$



Hero of Alexandria 10-70 AD