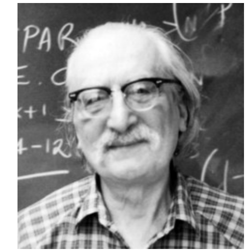
Shank series summation $L(n)=\dfrac{s(n+1)s(n-1)-s^2(n)}{s(n-1)+s(n+1)-2s(n)}$



Daniel Shanks
(1917-1996)

If the cumulative sum $s(n)$ of a sequence $x(n)$ (that is functionally dependent on integers n = 1,2....) converges to value S_∞, the idea is to suggest a functional form $f(n)$ of the difference between $s(n)$ and S_∞, and compute this for a modest number of terms. Since numeric computation of the series sum suggests S_∞ may not known *a priori*, (but we have strong evidence to suggest that the sequence sum converges), we can write

$$s(n)=L(n)+AB^n$$

Where $f(n)=AB^n$. We might anticipate exponential convergence between $s(n)$ and S_∞, so this is the motivation behind this functional form. The idea is that $L(n)$ tends to S_∞ as n tends to infinity, and $f(n)$ tends to zero.

$$\lim_{n\to\infty} L(n)=s_\infty, \quad \lim_{n\to\infty} AB^n=0.$$

Ideally $L(n)$ is a better predictor (i.e. converges more quickly) than $s(n)$, so could be evaluated to estimate S_∞ using smaller values of n than $s(n)$ directly.

$$L(n)=\frac{s(n+1)s(n-1)-s^2(n)}{s(n-1)+s(n+1)-2s(n)}$$

Since this method can be applied to any sequence, it could be applied to achieve efficient evaluation of functions which can be expressed by an infinite polynomial of sequentially decreasing magnitude terms (e.g. via a Maclaurin or Taylor series), or indeed as a method for evaluating piecewise-linear numeric integrals,* where the sequence relates to a geometric reduction of step size. (However, in the latter case, performance gains are often better achieved using 'rolling' higher order curve-fitting techniques such as *Simpson's rule* (quadratics) or *cubic splines* which fit polynomials to adjacent (x,y) data points. Once the polynomial coefficients are found, integration (and indeed differentiation) is easy to compute.

*i.e. a 'trapezium rule.'

Derivation of Shank's 'transform' i.e. series sum estimator

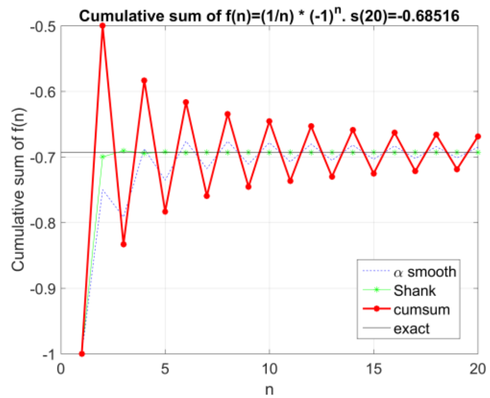
$$s(n)=\sum_{k=1}^{n}x(n), \quad s(n)=L(n)+AB^n$$

Definitions

$$L(n-1)\approx L(n)\approx L(n+1)\approx s_\infty$$

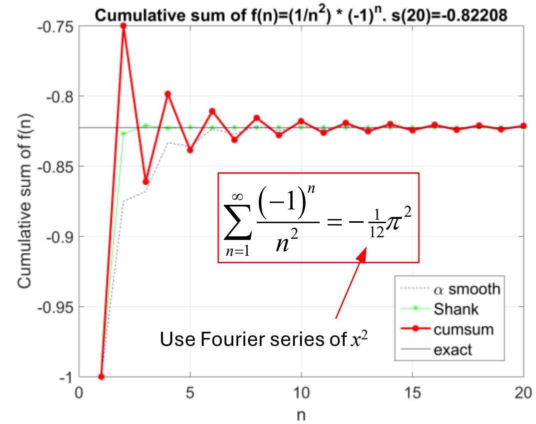
As n becomes large Assume L tends to a constant, which is the convergence of the sum $s(n)$.

$$\therefore s(n-1)-L=AB^{n-1}$$
$$\therefore s(n)-L=AB^{n}$$
$$\therefore s(n+1)-L=AB^{n+1}$$
$$\therefore \frac{s(n)-L}{s(n-1)-L}=B$$
$$\therefore \frac{s(n+1)-L}{s(n)-L}=B$$
$$\Rightarrow \frac{s(n)-L}{s(n-1)-L}=\frac{s(n+1)-L}{s(n)-L}$$
$$\therefore \left(s(n)-L\right)^2=\left(s(n+1)-L\right)\left(s(n-1)-L\right)$$
$$s^2(n)-2s(n)L+L^2=s(n+1)s(n-1)-L(s(n-1)+s(n+1))+L^2$$
$$s^2(n)-s(n+1)s(n-1)=L\{2s(n)-s(n-1)-s(n+1)\}$$
$$\therefore L=\frac{s^2(n)-s(n+1)s(n-1)}{2s(n)-s(n-1)-s(n+1)}$$
$$\therefore L(n)=\frac{s(n+1)s(n-1)-s^2(n)}{s(n-1)+s(n+1)-2s(n)}$$



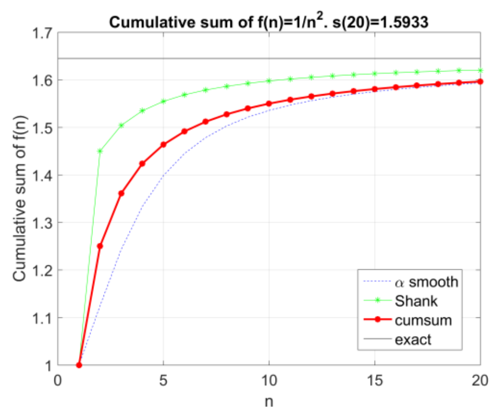
$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=-\ln 2$$

See Maclaurin expansion of $\ln(1+x)$ where $x=1$.



$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\tfrac{1}{12}\pi^2$$

Use Fourier series of x^2

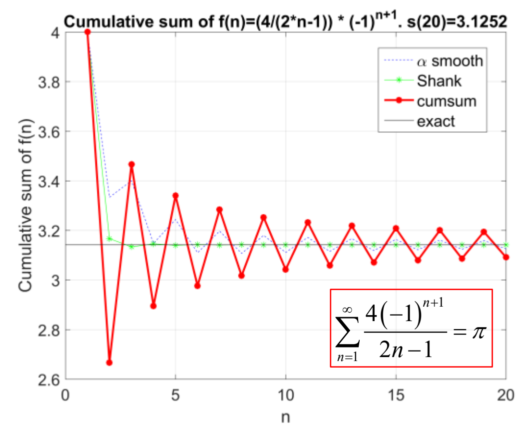


$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\tfrac{1}{6}\pi^2$$

Basel problem. Can solve by computing Fourier series of x.

α = 0.5 used in all examples

```
%Alpha smoothing
function y = alphasmooth(x,a)
y = x;
for n=2:numel(x)
    y(n) = (1-a)*y(n-1) + a*x(n);
end
```



$$\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{2n-1}=\pi$$