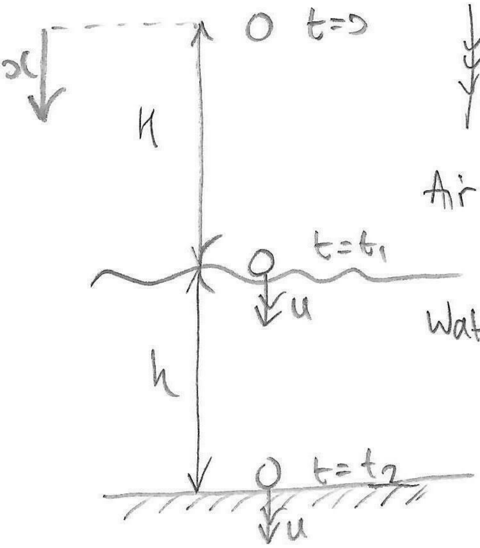
BPhO ROUND 1 SECTION 1 2024. AF SOLUTIONS.

a) A stone is dropped from rest from a height of 5.0 m (H) above the water surface of a pool. After hitting the water, it sinks to the bottom, 4.0 m below the surface. If it sinks with the same speed as it struck the water, for how long is the stone in motion? **[3]**

u $\quad h \quad t_2$



$g = 9.81\ \mathrm{m/s^2}$

Ignoring air resistance ⇒ constant acceleration motion in air.

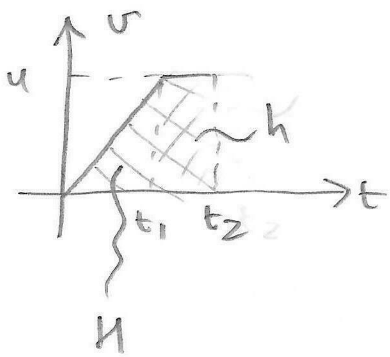
$$\boxed{\begin{aligned} H &= \tfrac{1}{2} g t_1^2 \\ u &= g t_1 \end{aligned}}$$

Assuming constant speed sinking at u m/s

$$\boxed{h = u(t_2 - t_1)}$$

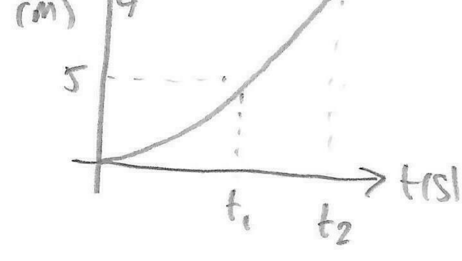
So $\quad t_1 = \sqrt{\dfrac{2H}{g}} \qquad u = \sqrt{2gH} \qquad t_2 = \dfrac{h}{u} + t_1$

$$\therefore \boxed{t_2 = \frac{h}{\sqrt{2gH}} + \sqrt{\frac{2H}{g}}}$$



$$\therefore t_2 = \frac{4.0}{\sqrt{2 \times 9.81 \times 5.0}} + \sqrt{\frac{2 \times 5.0}{9.81}} \ \ (\mathrm{s})$$

$$= \boxed{1.41\,\mathrm{s}} \ \text{to 3.s.f}$$

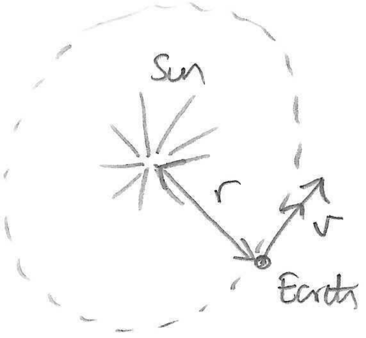


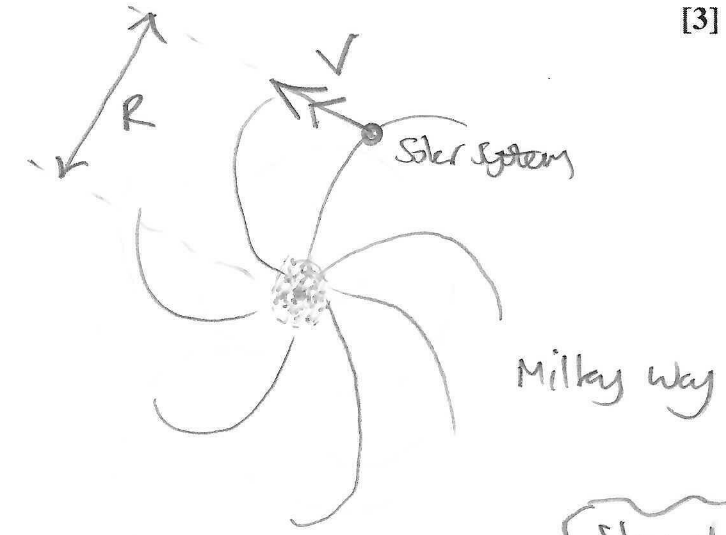
$[t_1 = 1.01\,\mathrm{s}, \quad u = 9.9\ \mathrm{m/s}]$

b) The Earth orbits the Sun at a distance of 8.0 light minutes away. The Sun lies at a position in the Milky Way galaxy at about 8.3 kpc from the centre. The galaxy rotates about the galactic centre with an orbital period of 240 million years. What is the value of the orbital speed of the Solar System in the galaxy divided by the orbital speed of the Earth about the Sun?

1.00 pc (parsec) = 3.26 light years

[3]





[orbital period = 1 Yr]

Stay algebraic! This includes the units

$$\frac{V}{v} = \frac{\cancel{2\pi} R / T}{\cancel{2\pi} r / \text{Yr}} = \frac{R}{r}\left(\frac{\text{Yr}}{T}\right)$$

$$= \frac{8.3 \times 10^3 \times 3.26 \text{ LYr}}{\left(\dfrac{8.0 \text{ LYr}}{365.25 \times 24 \times 60}\right)} \left(\frac{\text{Yr}}{240 \times 10^6 \text{ Yr}}\right)$$

[1 Yr = 365.25 × 24 × 60 minutes

so 1 light-minute = $\dfrac{1 \text{ LYr}}{365.25 \times 24 \times 60}$]

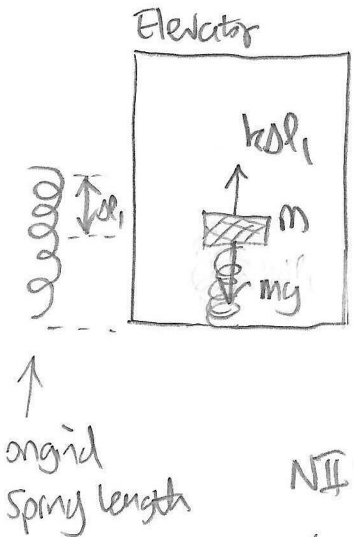
$$\therefore \frac{V}{v} = \frac{8.3 \times 10^3 \times 3.26}{8.0 \times 240 \times 10^6} \times 365.25 \times 24 \times 60$$

$$= \boxed{7.4}$$

[$v \simeq 29.8$ km/s, so $V \simeq 221$ km/s (!)]

c) A spring with spring constant, k, sits on the floor of of an elevator with a mass m resting on top. The elevator then accelerates upwards and uniformly from zero to $18\,\mathrm{m\,s^{-1}}$ in 6 s. What is the ratio $\frac{\Delta\ell_1}{\Delta\ell_2}$, where $\Delta\ell_1$ is the initial contraction in length of the spring when the lift is stationary, and $\Delta\ell_2$ is the contraction measured when the lift is accelerating?

[3]

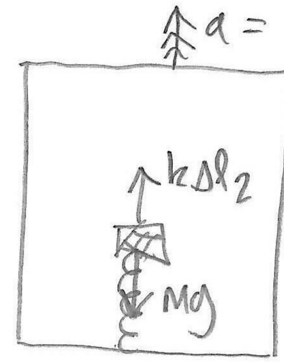


$\downarrow g = 9.81$ N/kg

STATIONARY

$\uparrow a = 18/6\ \mathrm{m/s^2}$

ie $a = 3\,\mathrm{m/s^2}$



{ Assume relative motion to elevator → 0 ie Δl_2 fixed }

NII: $0 = k\Delta l_1 - mg$

$\therefore \boxed{\Delta l_1 = \frac{mg}{k}}$

NII $\uparrow$: $Ma = k\Delta l_2 - Mg$

$\therefore \boxed{\Delta l_2 = \frac{M(a+g)}{k}}$

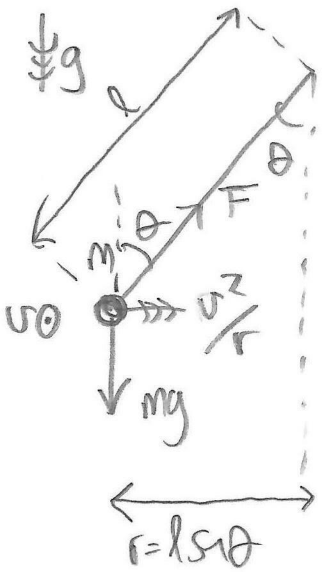
So $\frac{\Delta l_1}{\Delta_2} = \frac{g}{a+g} = \frac{9.81}{3.0 + 9.81} = \boxed{0.77}$

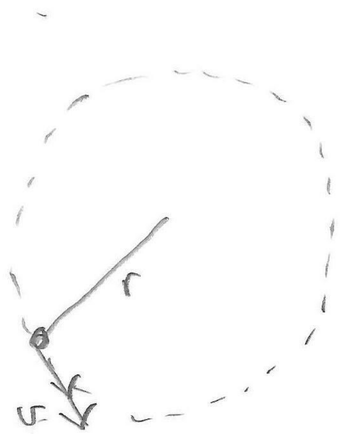
d) A mass m hangs from the end of a light string of length ℓ, which is attached to a point on the ceiling. It is set in motion so that the mass moves in a horizontal circle of radius r with the string at angle θ to the vertical as it traces out the curved surface of a cone.

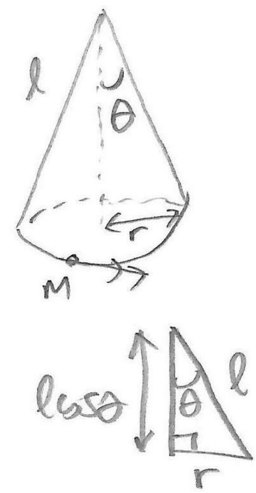
(i) By resolving the forces on the mass, obtain an expression for the period of orbit, T, of the mass, in terms of r, g and $\tan\theta$.

(ii) Substitute for $\tan\theta$ to express the period of orbit in terms of ℓ, g and $\cos\theta$.

[4]







(i) (let F be string tension since T used for period (:))

NII: ↑: $0 = F\cos\theta - mg \Rightarrow \boxed{F = mg/\cos\theta}$.

radially inwards →: $\dfrac{mv^2}{r} = F\sin\theta$

$\therefore \dfrac{\cancel{m}v^2}{r} = \dfrac{\cancel{m}g}{\cos\theta}\sin\theta \Rightarrow \boxed{v = \sqrt{rg\tan\theta}}$

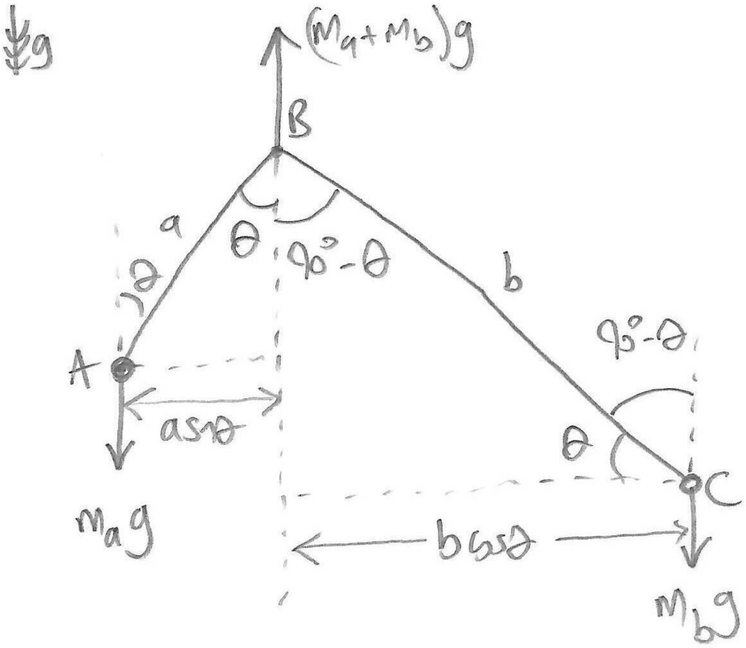
Now $v = \dfrac{2\pi r}{T}$ so $T = 2\pi \dfrac{r}{v}$

$\therefore \boxed{T = 2\pi\sqrt{\dfrac{r}{g\tan\theta}}}$

(ii) $\tan\theta = \dfrac{r}{\ell\cos\theta}$ $\therefore$ $T = 2\pi\sqrt{\dfrac{\cancel{r}}{g}\dfrac{\ell\cos\theta}{\cancel{r}}} = \boxed{2\pi\sqrt{\dfrac{\ell\cos\theta}{g}}}$

e) A light lever in the form of a letter L, with arms AB, BC of lengths a, b, is smoothly pivoted at B so that it can turn in a vertical plane. Masses m_a and m_b are attached at A and C respectively. In an equilibrium position, the inclination of AB to the vertical is given by angle θ. Obtain an expression for θ in term of m_a, m_b, a and b.

[4]



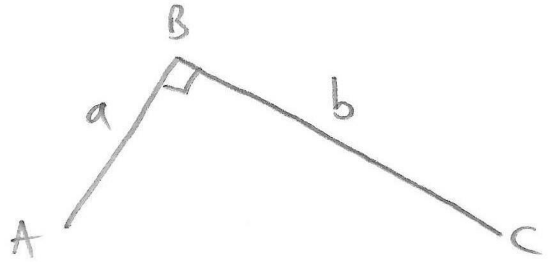
Moments ↻+ about B

$0 : M_b g\, b\cos\theta - M_a g\, a\sin\theta$

Eq

$\therefore \dfrac{\sin\theta}{\cos\theta} = \dfrac{M_b b}{M_a a}$

$\therefore \boxed{\tan\theta = \left(\dfrac{M_b}{M_a}\right)\left(\dfrac{b}{a}\right)}$

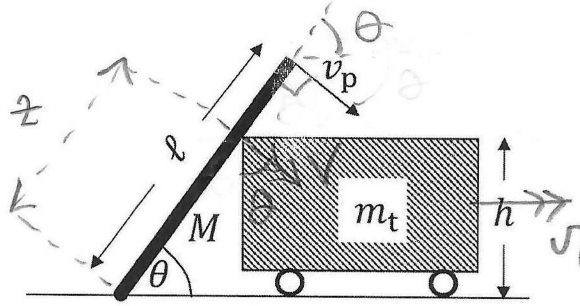


light lever
(ignore masses of arms)

f) A heavy uniform plank of length ℓ and mass M, is fixed to the ground at its lower end by a smooth hinge. Initially, it stands upright next to a stationary trolley of mass m_t and height h, as shown in **Fig. 1**. The plank is disturbed and it tips over, pushing the trolley along horizontally with the smooth flat surface of the plank pushing the top rear corner of the trolley.
When the plank is falling over, the top end swings in a circle at speed v_p, such that the plank's rotational kinetic energy about the pivot at the lower fixed end is given by $\frac{1}{6}Mv_p^2$.
At the moment the top of the plank reaches the top corner of the trolley, obtain an expression for the speed of the trolley, v_t, in terms of g and h, given $M = \frac{m_t}{2}$ and $\ell = 3h$.
Neglect friction in this question.

Trolley moving at $v\sin\theta = v_t$

$\frac{v}{v_p} = \frac{z}{\ell}$



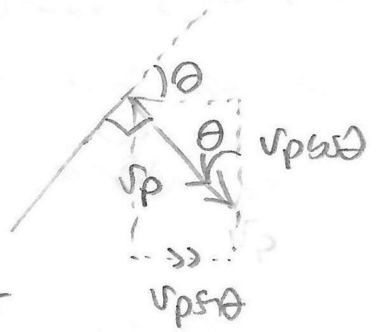


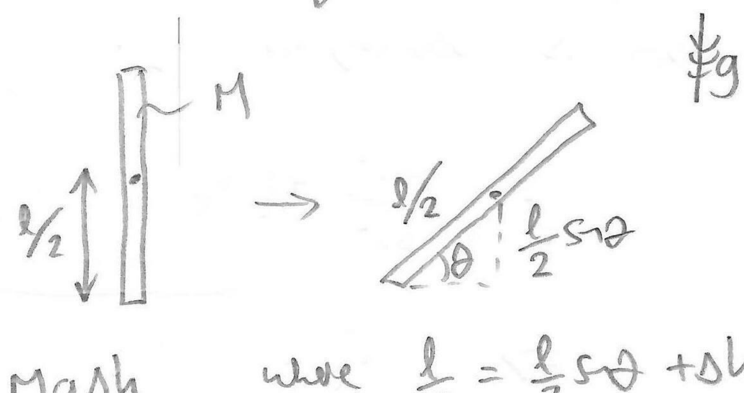
Figure 1: A smooth plank falling over and pushing the top corner of a trolley along a frictionless horizontal surface.

$h = z\sin\theta \Rightarrow v = v_p \frac{h}{\sin\theta}\frac{1}{\ell} \to v = \frac{v_p h}{\ell\sin\theta}$ $\therefore$ $\boxed{v_t = \frac{v_p h}{\ell}}$

[4]

Start with conservation of energy:

$$\underbrace{\frac{1}{6}Mv_p^2}_{\text{KE of plank}} + \underbrace{\frac{1}{2}M_t v_t^2}_{\text{KE of trolley}} = \text{GPE loss of plank}$$



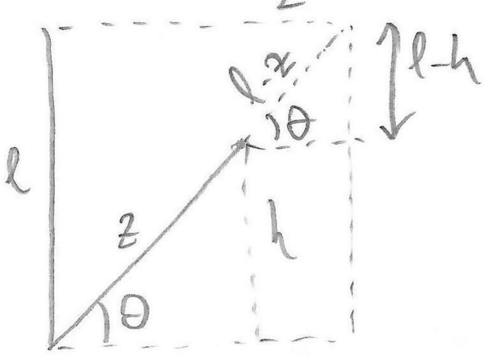
GPE loss of plank is $Mg\delta h$ where $\frac{\ell}{2} = \frac{\ell}{2}\sin\theta + \delta h$

$\therefore$ GPE loss is $\boxed{Mg\frac{\ell}{2}(1-\sin\theta)}$.

Now assume plank and trolley are in contact when plank top has speed v_p. $\therefore$ $\boxed{v_t = v_p h/\ell}$ {ie v_p is defined "post inelastic collision" - possible gotcha if you think this is a Before and After collision problem involving conservation of momentum}

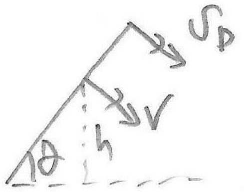
So $v_p = \frac{l v_t}{h}$

$\therefore$ energy expression is: $\frac{1}{6} M \frac{l^2 v_t^2}{h^2} + \frac{1}{2} M_t v_t^2 = Mg\frac{l}{2}(1-\sin\theta)$



* MS says $v_p \sin\theta = v_t$, but I'm not sure this is correct.

$v_t = V\sin\theta$ and $V = \frac{v_p h}{l\sin\theta}$



So $v_t^2 \left(\frac{Ml^2}{6h^2} + \frac{1}{2}M_t\right) = \frac{Mgl(1-\sin\theta)}{2}$

Since tip of plank drops $l-h$, might be easier
* simply to write ΔGPE as $\frac{1}{2}Mg(l-h)$ since *
CoM is originally at $l/2$.

$\therefore \quad v_t^2 \left(\frac{Ml^2}{6h^2} + \frac{1}{2}M_t\right) = \frac{1}{2}Mg(l-h)$

$$\Rightarrow \boxed{v_t = \sqrt{\frac{Mg(l-h)}{\frac{Ml^2}{3h^2} + M_t}}}$$

If $l = 3h$ and $M = M_t/2 \Rightarrow M_t = 2M$

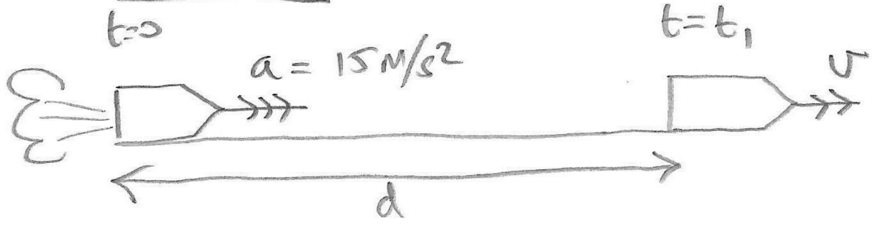
$$\Rightarrow v_t = \sqrt{\frac{Mg(2h)}{\frac{9Mh^2}{3h^2} + 2M}} = \boxed{\sqrt{gh}\sqrt{\frac{2}{5}}}$$

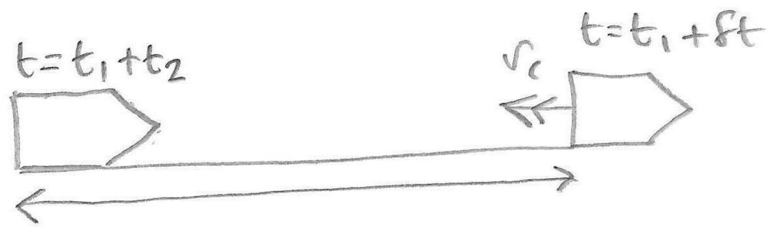
a

g) A rocket testing machine consists of a track of length d, along which a rocket can accelerate uniformly from rest at 15 m s^{-2} up to the end point, where a set of brakes rapidly brings it to rest in a negligible time. It is then drawn back to the start at a constant speed of $v_c = 12\ \mathrm{m\,s^{-1}}$. If the time taken from firing the rocket to its return to the starting point is 48 s, what is the length of the track, d?

$\boxed{T = t_1 + t_2}$

[6]





$[\delta t \ll 1s \text{ so } \approx 0]$

$d = \frac{1}{2}at_1^2 \Rightarrow \sqrt{2d/a} = t_1$

$d = v_c t_2 \Rightarrow d/v_c = t_2$

$\therefore T = \sqrt{\frac{2d}{a}} + \frac{d}{v_c}$

$\left(T - \frac{d}{v_c}\right)^2 = \frac{2d}{a}$

$T^2 - \frac{2Td}{v_c} + \frac{d^2}{v_c^2} - \frac{2d}{a} = 0$

$v_c^2T^2 - 2Tdv_c + d^2 - \frac{2d}{a}v_c^2 = 0$

$d^2 - d\left(2Tv_c + \frac{2v_c^2}{a}\right) + v_c^2T^2 = 0$

$\left(d - Tv_c - \frac{v_c^2}{a}\right)^2 - \left(Tv_c + \frac{v_c^2}{a}\right)^2 + v_c^2T^2 = 0$

$\therefore d = \pm\sqrt{\left(Tv_c + \frac{v_c^2}{a}\right)^2 - v_c^2T^2} + Tv_c + \frac{v_c^2}{a}$

$$\boxed{\therefore d = \pm\sqrt{\frac{2Tv_c^3}{a} + \frac{v_c^4}{a^2}} + v_c\left(T + v_c/a\right)}$$

Better: $d = v_c\left[T + \frac{v_c}{a} \pm \sqrt{\frac{2Tv_c}{a} + \frac{v_c^2}{a^2}}\right]$

+ root: $d = 691.2\ \mathrm{m}$

$t_1 + t_2 = 67.2\,s \neq 48\,s$ ✗

− root: $\boxed{d = 480\ \mathrm{m}}$

$t_1 + t_2 = 48\,s$ ✓

$a = 15\ \mathrm{m/s^2}$

$v_c = 12\ \mathrm{m/s}$ $T = 48\,s$

h) The displacement y of a point in an elastic medium when a wave passes through depends on the time t and the location x of the point, relative to the starting time and the origin. The displacement is obtained from the phase of the wave at that point, which is the sum of the phases due to the time increment and the relative distance from the origin. The displacement of a point at x and t, is given by

$$y = 4.0 \times 10^{-6}(\text{m}) \cos\left(1800\frac{t}{(\text{s})} + 5.72\frac{x}{(\text{m})}\right)$$

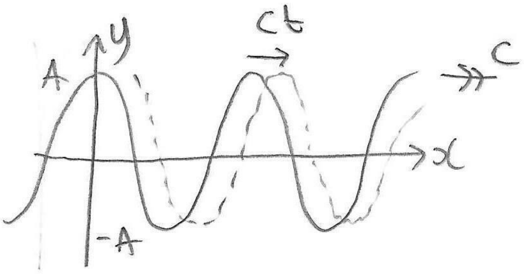
(Units are given in brackets)

A f λ c

(i) Determine the amplitude, frequency, wavelength, and speed of the wave.

(ii) What is the maximum speed of motion of the point in the medium?

[4]



$$y = A\cos\left(\frac{2\pi(x-ct)}{\lambda} - \phi\right)$$

initial phase

So ~~y~~ $y = 4.0\times10^{-6}\cos(1800t + 5.72x)$

(i) $A = 4.0\times10^{-6}$ (m) $\qquad \phi = 0.$

$$\frac{2\pi x}{\lambda} - \frac{2\pi ct}{\lambda} = 1800t + 5.72x$$

$$\therefore \quad \frac{2\pi}{\lambda} = 5.72 \Rightarrow \lambda = \frac{2\pi}{5.72} \text{ (m)} \Rightarrow \lambda = \boxed{1.1\text{ m}}$$

$$-\frac{2\pi c}{\lambda} = 1800 \Rightarrow |c| = 1800\times\frac{2\pi}{5.72}\frac{1}{2\pi} = \boxed{315\text{ m/s}}$$

[Note wave moves right to left as stated ie $c = -315$ m/s]

$$|c| = f\lambda \Rightarrow f = 315/1.1 = \boxed{286\text{ Hz}}$$

(ii) $$\frac{\partial y}{\partial t} = -1800\times4.0\times10^{-6}\sin(1800t + 5.72x)$$

So $$\left|\frac{\partial y}{\partial t}\right|_{\text{max}} = 1800\times4.0\times10^{-6} = \boxed{7.2\times10^{-3}\text{ m/s}}$$

Note: $\boxed{\text{if } \; y = A\cos(kx - \omega t - \phi)}$

$$k = 2\pi/\lambda \quad , \quad \omega = 2\pi f = \frac{2\pi c}{\lambda}$$

$$\frac{\partial y}{\partial t} = + \omega A \sin(kx - \omega t - \phi)$$

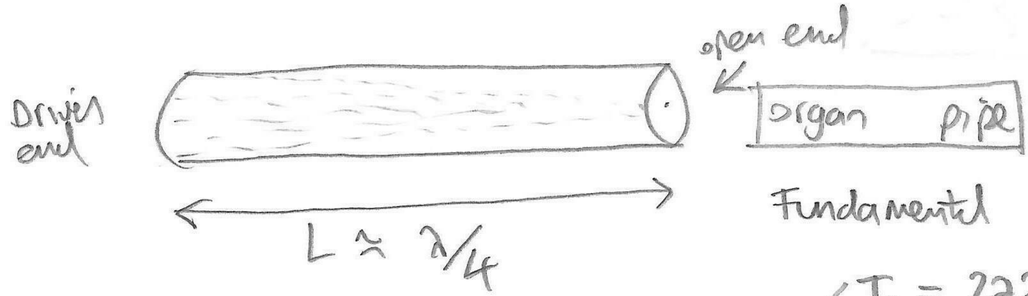
so $$\left|\frac{\partial y}{\partial t}\right|_{max} = \omega A = \boxed{\frac{2\pi c}{\lambda} A}$$

i) Beats are produced when two waves of similar frequencies interfere, causing slowly varying constructive and destructive interference. If the frequencies differ by 1 Hz then the constructive to destructive to constructive interference will occur once each second, giving a beat frequency of 1 Hz.

A siren producing a note of 256 Hz and an organ pipe of slightly lower frequency full of air at 17 °C, when played together give a beat frequency of 4 Hz. On slowly changing the temperature of the air in the pipe, the beat frequency first reduces to zero and then increases again to 4 Hz. what is the change in temperature of the air in the pipe?

The speed of a wave in air is given by $v = \sqrt{\frac{\gamma P}{\rho}}$ where γ is a numerical constant, P is the pressure and ρ is the density of the air.

[4]



Fundamental standing wave.

g pressure in this case

ie node at open end

$T_1 = 273 + 17$ K

* Assume organ frequency at 17°C is $(256 - 4)$ Hz $= f_1$
and " " at temp T_2 is $(256 + 4)$ Hz $= f_2$

* Now if air in pipe is an ideal gas $\boxed{PV = nRT}$

$n = \frac{m}{M_{mol}}$ ← mass of air ; M_{mol} ← molar mass of air molecules

$\Rightarrow P = \frac{m}{V} \frac{RT}{M_{mol}}$; $\frac{m}{V}$ ← density ρ

So $P/\rho = \frac{R}{M_{mol}} T$

[$R = 8.314$ J mol^{-1} K^{-1}
$M_{mol} \simeq 28.97$ g/mol]

$\therefore v = \sqrt{\frac{\gamma R}{M_{mol}} T} \Rightarrow \boxed{v \propto \sqrt{T}}$

$v = f\lambda$. If λ is the same for the pipe (ie $L = \lambda/4$ and assume negligible ~~thermal~~ expansion),

let $\boxed{v = k\sqrt{T}} \Rightarrow \frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}} \quad \therefore \boxed{T_2 = T_1 \left(\frac{f_2}{f_1}\right)^2}$

$$\therefore \quad \Delta T = T_2 - T_1$$

$$= T_1 \left(\left(\frac{f_2}{f_1} \right)^2 - 1 \right)$$

$$= 290 \left(\left(\frac{260}{252} \right)^2 - 1 \right) \quad k \quad \text{(or °C since a change)}$$

$$= \boxed{18.7\,°C}$$

$\swarrow T_0\ ^\circ C$

j) Pure water can be cooled down to $-10\,^\circ C$ without freezing. If a small piece of ice is dropped into this supercooled water held in a thermally insulated container, some of the water immediately freezes. What fraction will freeze?

$L_{ice} = 335\,000\,J\,kg^{-1}$

$c_{water} = 4180\,J\,kg^{-1}\,^\circ C^{-1}$

$c_{ice} = 2070\,J\,kg^{-1}\,^\circ C^{-1}$

↑Need this really.

[Sort of !]

[3]

Energy balance idea: * Formation of ice from liquid water will release energy, warming the water up. * But this process can only occur when overall temperature {assume water and ice in ~~thermal~~ equilibrium} $\leq 0^\circ C$. Above this none of the water can form ice, without a supercooling process!

So in extreme of ΔT of water is T_0 of $10^\circ C$:

ie no more ice can be formed, and if $T > 0^\circ C$ ice will start to melt.

$$\underbrace{M_{ice} L_{ice}}_{\text{Energy released from freezing } M_{ice} \text{ kg}} = \underbrace{c_W M_W T_0}_{\text{Energy that raises remaining water by } T_0} + \underbrace{c_{ice} M_{ice} T_0}_{\text{raise ice to } 0^\circ C}$$

Now $c_{ice} \simeq 2070\,J\,kg^{-1}\,^\circ C^{-1}$, but this isn't given in the question.

MS states: $\dfrac{M_{ice}}{M_W} = \dfrac{c_W T_0}{L_{ice}} = \dfrac{4180 \times 10}{335000} \simeq \boxed{12\%}$

which implies the ice remains at $-10^\circ C$. But in this case the 'fraction that freezes' ought to apply to the original mass of water, so $\dfrac{M_{ice}}{M_{ice} + M_W} = \dfrac{1}{M_{ice}/M_W + 1} = \dfrac{1}{\frac{1}{0.12} + 1} = \boxed{11\%}$

Note if the ice is at 0°C

$$\frac{M_{ice}}{M_w} = \frac{C_w T_0}{L_{ice} - C_{ice} T_0} = \frac{4180 \times 10}{335000 - 2070 \times 10}$$

$$= 0.133$$

$$\therefore \frac{M_{ice}}{M_w + M_{ice}} = \frac{M_{ice}/M_w}{1 + M_{ice}/M_w} = \frac{0.133}{1 + 0.133} = \boxed{0.12}$$

So 12% is actually the 'best' answer, but I suggest the question needs C_{ice} and the MS ratio $\frac{M_{ice}}{M_w}$ is not quite correct.

k) A circuit with an ideal diode in series with a resistor R is shown in **Fig. 2**. Two supply voltages each with amplitude V_0 and frequency $f = \frac{1}{T}$ are also shown: in (a) as a square wave, and (b) a sinusoidal variation with time. Obtain expressions for the average power dissipated in the resistor R for

$\langle P \rangle$

(i) the square wave signal shown in (a), and

(ii) for the sinusoidal variation shown in (b).

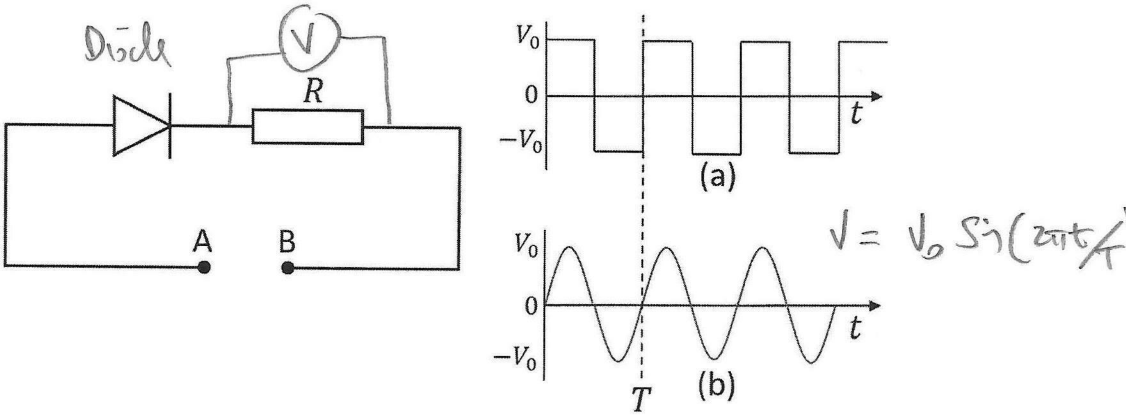
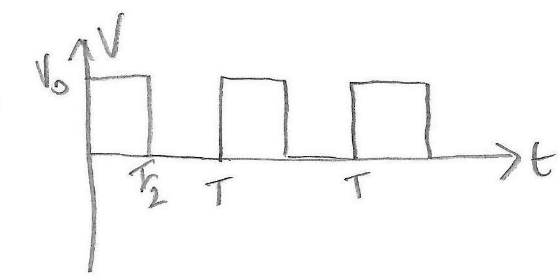


Figure 2: A diode in series with a resistor R, with two a.c. voltage supplies (a) a square wave, and (b) a sinusoidal time dependence.

[4]

* Resistance of diode $\simeq \infty$ (Ω) when −ve voltage applied, $\simeq 0\ \Omega$ when +ve voltage applied.

* So



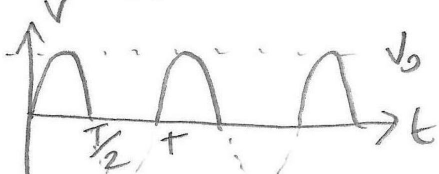
for Square wave (across R).

$P = V^2/R$ so $\langle P \rangle = \dfrac{\langle V^2 \rangle}{R}$ ie average over one period T.

$$\langle V^2 \rangle = \frac{\int_0^T V^2 dt}{T} = \frac{V_0^2 \, T/2}{T} = \frac{V_0^2}{2}$$

So $\langle P \rangle = \boxed{\dfrac{V_0^2}{2R}}$

* for



$$\langle V^2 \rangle = \frac{\int_0^T V_0^2 \sin^2(2\pi t/T)\, dt}{T} = \frac{\frac{V_0^2}{2}\left(\frac{T}{2}\right)}{T}$$

Symmetry. You don't need to actually evaluate the integral!

$$\Rightarrow \langle V^2 \rangle = V_0^2/4 . \qquad \text{So } \langle P \rangle = \frac{\langle V^2 \rangle}{R} = \boxed{\frac{V_0^2}{4R}}$$

1) A uniform wire of resistance per unit length r is formed into a square of side ℓ, with diagonals that are connected at the centre. This is shown in **Fig. 3**.
Obtain an expression for the resistance between points A and B at the corners of the square, in terms of r and ℓ. Give you answer in surds.

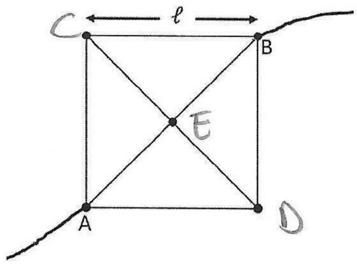
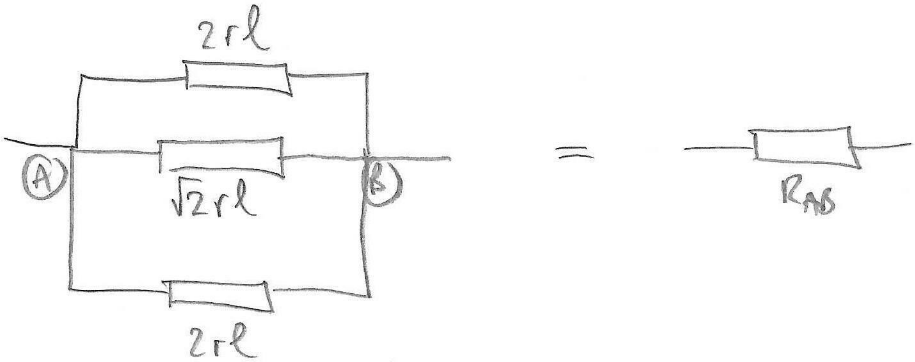


Figure 3: Wires connected to form a square with diagonals, with connection leads at A and B.

[3]

C, D are equipotentials, so no current flows CED.

So equivalent circuit is:



$$R_{AB} = \frac{1}{\frac{1}{2r\ell} + \frac{1}{2r\ell} + \frac{1}{\sqrt{2}r\ell}} = \frac{r\ell}{1 + \frac{1}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}}{\sqrt{2}+1} r\ell$$

$$= \frac{\sqrt{2}(1-\sqrt{2})}{(\sqrt{2}+1)(-\sqrt{2}+1)} r\ell$$

$$= \frac{\sqrt{2}-2}{-2+1} r\ell$$

$$= \boxed{(2-\sqrt{2})r\ell}$$

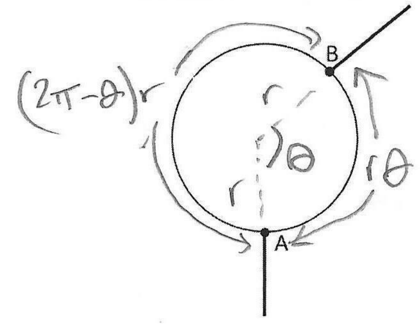
$A = \pi\left(\frac{d}{2}\right)^2$ ρ

m) A length of resistance wire of diameter 1.4 mm and resistivity $3.0 \times 10^{-4}\,\Omega\,\mathrm{m}$ is bent to form a circle, as shown in **Fig. 4**. The radius of the circle is 10 cm (r). Two conducting wires are connected at points A and B, such that the contact B can slide around the circle.

(i) Calculate the maximum resistance that can exist between A and B.

(ii) Sketch a graph of the resistance between A and B against the arc length measured form A as contact B is slid from A around the circle and back to A on the other side.

(iii) What is the shortest length of the arc between A and B such that the resistance between A and B is $22.0\,\Omega$?



θ in radians

Figure 4: A wire bent into the shape of a circle with contacts at A and B. B can slide around the circle.

[6]

(i), (ii)

$$R_{AB} = \frac{1}{\frac{1}{\frac{\rho r\theta}{A}} + \frac{1}{\frac{\rho(2\pi-\theta)r}{A}}}$$

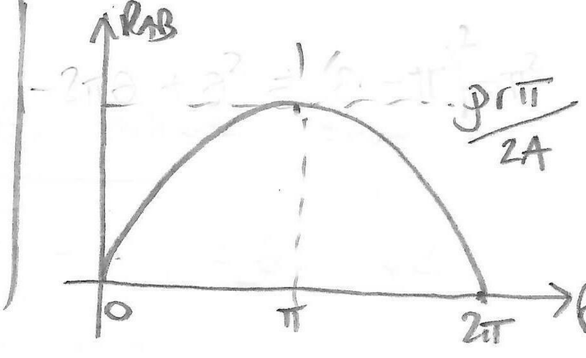
$$= \frac{\rho r}{A}\left(\frac{1}{\frac{1}{\theta}+\frac{1}{2\pi-\theta}}\right)$$

$$= \frac{\rho r}{A}\left(\frac{(2\pi-\theta)\theta}{2\pi-\theta+\theta}\right)$$

[A, L

$R = \frac{\rho L}{A}$]

$\therefore$ $\boxed{R_{AB} = \frac{\rho r}{2\pi A}\left(2\pi\theta - \theta^2\right)}$



Max R_{AB} when $\theta = \pi$

$\Rightarrow R_{AB} = \frac{\rho r}{2\pi A}\pi^2 = \boxed{\frac{\rho r \pi}{2A}}$

$\boxed{R_{AB} = \frac{\rho r}{2\pi A}\theta(2\pi-\theta)}$

↑ most useful form

If $d = 1.4\times10^{-3}$ m, $\rho = 3.0\times10^{-4}\,\Omega$m, $r = 0.1$ m

$$\max R_{AB} = \frac{\rho r\pi}{2A} = \frac{3.0\times10^{-4}\times0.1\times\pi}{2\times\pi\left(\frac{1.4}{2}\times10^{-3}\right)^2}\ \Omega$$

$\left[A = \pi\left(\frac{d}{2}\right)^2 = 1.54\times10^{-6}\,\text{m}^2\right]$

$$= \boxed{30.6\,\Omega} = 31\,\Omega \text{ to 2 s.f.}$$

(iii) let $R_{AB} = 22\,\Omega$ want $\boxed{r\theta}$

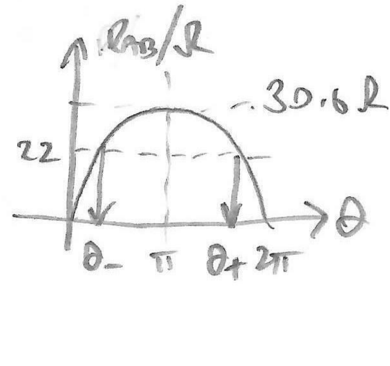
$$R_{AB} = \frac{\rho r}{2\pi A}\theta(2\pi-\theta)$$

$$\Rightarrow \frac{2\pi A R_{AB}}{\rho r} = 2\pi\theta - \theta^2$$

$$\Rightarrow \theta^2 - 2\pi\theta + k = 0 \qquad k = \frac{2\pi A R_{AB}}{\rho r}$$

$$\Rightarrow (\theta-\pi)^2 - \pi^2 + k = 0$$

$$\therefore \boxed{\theta = \pm\sqrt{\pi^2-k} + \pi}$$



We want shortest $r\theta$,

$$\text{so } r\theta_- = \boxed{\left(\pi - \sqrt{\pi^2-k}\right)r}$$

$k = 7.093$ so if $r = 0.1$, $r\theta = \boxed{0.148\text{ m}}$

so smallest arc is 14.8 cm giving a total resistance of $22\,\Omega$.

n) Before a device is fully charged, the current flowing into it at a time $t < \underbrace{t_{\text{fully charged}}}_{T}$ is given by

$$I = I_0 - \alpha t$$

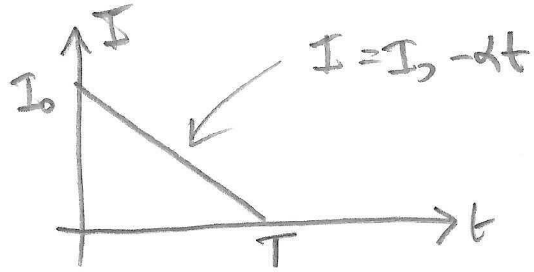
where I_0 is the initial current.

(i) How long does it take to charge it, and

(ii) how much charge is accumulated?

[4]

(i)



$I = 0$ when

$I_0 - \alpha T = 0$

$\Rightarrow \boxed{T = I_0/\alpha}$

(ii) $I = \dfrac{dQ}{dt}$ so $Q = \displaystyle\int_0^T I\,dt$

ie area under (I,t) graph. $\therefore$ $Q = \frac{1}{2} I_0 T$

$= \boxed{\frac{1}{2} I_0^2/\alpha}$

[Could also do by integration: $Q = \displaystyle\int_0^T (I_0 - \alpha t)\,dt$

$= \left[I_0 t - \frac{1}{2}\alpha t^2 \right]_0^T$

$= I_0 T - \frac{1}{2}\alpha T^2$

$= \dfrac{I_0^2}{\alpha} - \frac{1}{2}\alpha \dfrac{I_0^2}{\alpha^2}$

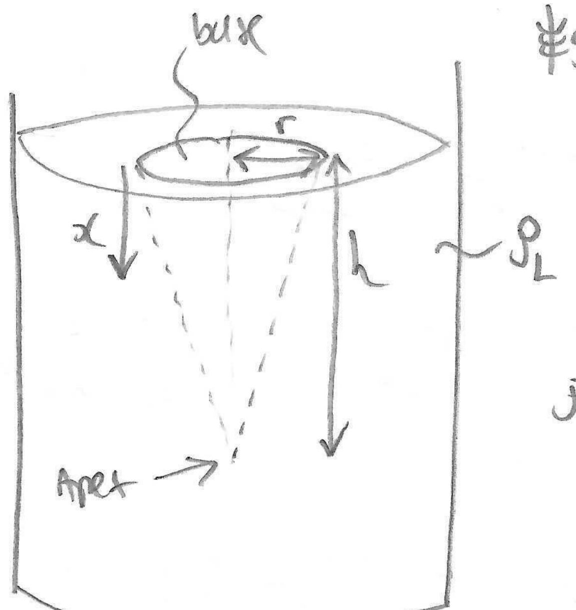
$= \boxed{\frac{1}{2} I_0^2/\alpha}$

Note MS states $\frac{\rho_a}{\rho_L} = \frac{5}{24}$. It should be $\boxed{\frac{5}{2}}$ Algebraically tedious!

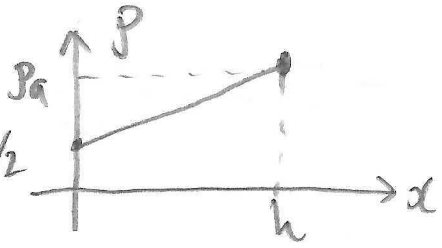
o) A cone of circular cross section sits in a liquid of density ρ_ℓ with apex pointing down and its base level with the surface of the liquid. The density of the cone varies linearly along its length. The base of the cone has a density ρ_b half that of the liquid. What is the ratio of the density of the apex, ρ_a, to the density of the liquid?

[6]

~~g~~



cone density with distance from base



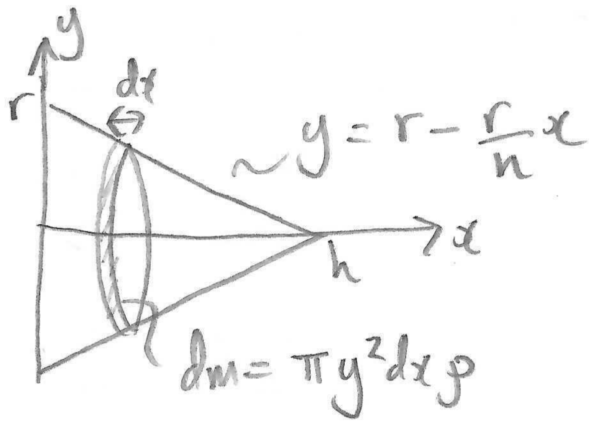
So $$\boxed{\rho = \frac{\rho_L}{2} + \frac{x}{h}\left(\rho_a - \frac{\rho_L}{2}\right)}$$

If cone is floating just at the surface $\Rightarrow$ weight = upthrust. ie weight of cone = weight of fluid displaced.

$\Rightarrow$ (since g is constant for a sensible cone height (!))

Mass of cone = mass of fluid displaced. $= \boxed{\frac{1}{3}\pi r^2 h \rho_L}$

↑ Volume of cone assumed



So cone mass is

$$M = \int_0^h dx\, \pi \left(r - \frac{r}{h}x\right)^2 \left(\frac{\rho_L}{2} + \frac{x}{h}\left(\rho_a - \frac{\rho_L}{2}\right)\right)$$

$$= \pi \int_0^h \left(r^2 - \frac{2r^2}{h}x + \frac{r^2}{h^2}x^2\right)\left(\frac{\rho_L}{2} + \frac{x}{h}\left(\rho_a - \frac{\rho_L}{2}\right)\right) dx$$

$$= \pi \int_0^h \left(r^2 \frac{\rho_L}{2} + x\left\{ -\frac{2r^2}{h}\frac{\rho_L}{2} + \frac{r^2}{h}\left(\rho_a - \frac{\rho_L}{2}\right) \right\} + x^2\left\{ \frac{r^2\rho_L}{2h^2} - \frac{2r^2}{h^2}\left(\rho_a - \frac{\rho_L}{2}\right) \right\} \right) dx$$

$$+ \pi \int_0^h \frac{r^2}{h^3}\left(\rho_a - \frac{\rho_L}{2}\right) x^3 \, dx$$

$$= \frac{\pi r^2 \rho_L h}{2} + \frac{1}{2}h^2\pi \left\{ -\frac{r^2\rho_L}{h} + \frac{r^2\rho_a}{h} - \frac{r^2\rho_L}{2h} \right\}$$

$$+ \frac{\pi}{3}h^3 \left\{ \frac{r^2\rho_L}{2h^2} - \frac{2r^2\rho_a}{h^2} + \frac{r^2\rho_L}{h^2} \right\} + \frac{\pi h^4}{4}\frac{r^2}{h^3}\left(\rho_a - \frac{\rho_L}{2}\right)$$

$$= \pi r^2 h \left\{ \frac{\rho_L}{2} - \frac{\rho_L}{2} + \frac{\rho_a}{2} - \frac{\rho_L}{4} + \frac{\rho_L}{6} - \frac{2}{3}\rho_a + \frac{\rho_L}{3} + \frac{\rho_a}{4} - \frac{\rho_L}{8} \right\}$$

$$= \pi r^2 h \left\{ \rho_L\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{1}{6} + \frac{1}{3} - \frac{1}{8}\right) + \rho_a\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) \right\}$$

$$= \pi r^2 h \left(\frac{\rho_L}{8} + \frac{\rho_a}{12} \right)$$

So if $\frac{1}{3}\cancel{\pi r^2 h}\rho_L = \cancel{\pi r^2 h}\left(\frac{\rho_L}{8} + \frac{\rho_a}{12}\right)$

$$\rho_L = \left(\frac{3}{8}\rho_L + \frac{\rho_a}{4}\right)$$

$$\frac{5}{8}\rho_L = \frac{\rho_a}{4}$$

$$\frac{20}{8} = \frac{\rho_a}{\rho_L}$$

$$\boxed{\frac{5}{2} = \frac{\rho_a}{\rho_L}}$$

MS ✓

$$\frac{\rho_L}{3} = \frac{\rho_L}{8} + \frac{\rho_a}{12}$$

$$\rho_L = \frac{3\rho_L}{8} + \frac{3\rho_a}{12}$$

$$\frac{5}{8}\rho_L = \frac{3\rho_a}{12}$$

$$\frac{5 \times 12}{8 \times 3} = \frac{\rho_a}{\rho_L}$$

$$\frac{5 \times 12}{24} = \frac{\rho_a}{\rho_L}$$

MS states $\frac{5}{24} = \frac{\rho_a}{\rho_L}$

↑ I think a mistake. Answer should be $\boxed{\frac{5}{2}}$.

p) An elastic material called neoprene is used within sections of motorway roads in hot countries to maintain a flat road with no discontinuities under different temperatures. If we have a road built with 2.0 cm of neoprene placed between every 20 m lengths of concrete, and the concrete has a thermal expansion coefficient of $\alpha = 12\times10^{-6}\,\mathrm{K}^{-1}$, what is the highest temperature it can go up to before the road cracks?

x_0 $\qquad$ l_0

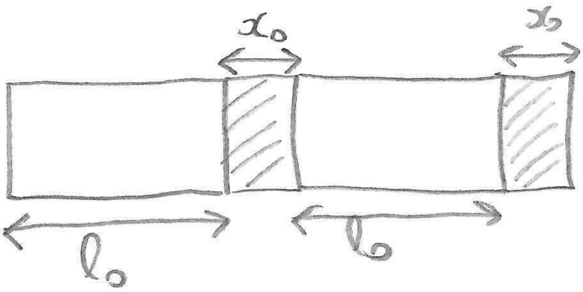
Assume that the elastic material is initially under no deformation at a temperature of 20 °C.

Thermal expansion coefficient α is specified by $\ell = \ell_0(1+\alpha\Delta T)$ in which ℓ is the expanded length, ℓ_0 the initial length, and ΔT the temperature rise.

Young's Modulus of neoprene = 2.7 MPa Y

Breaking stress of concrete = 30 MPa σ_B

[5]



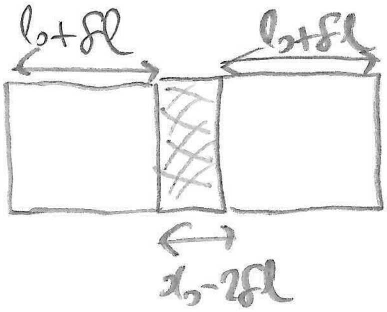
$x_0 = 2.0\times10^{-2}$ m

$l_0 = 20$ m

* If concrete expands due to temperature rise ΔT

$l \to l_0 + \delta l$ where $\boxed{\delta l = l_0 \alpha \Delta T}$

* The neoprene $\therefore$ is compressed by $\boxed{2\delta l}$ ← not δl.



* Assume concrete cracks when neoprene stress σ_n = breaking stress of concrete

Neoprene: $\quad Y = \dfrac{\sigma_n}{2\delta l / x_0} \quad \Rightarrow \quad \sigma_n = \dfrac{2}{x_0} l_0 \alpha \Delta T\, Y.$

So if $\sigma_B = \sigma_n \quad \Rightarrow \quad \Delta T = \dfrac{\sigma_B x_0}{2 l_0 \alpha Y}$

$\Rightarrow \Delta T = \dfrac{30\times10^{6}\times 2\times10^{-2}}{(2\times20)\times 12\times10^{-6}\times 2.7\times10^{6}} = \boxed{463^\circ\mathrm{C}}$

MS is factor of two too high, unless $l_0 \to 10\mathrm{m}$!

So if $T_0 = 20^\circ\mathrm{C}$

$\Rightarrow T > 483^\circ\mathrm{C}$ when road cracks.

[MS: $l_0 = 10$m not 20m]

q) A laser beam that entered the region between two infinite ideal mirrors would reflect an infinite number of times between them, as shown in **Fig. 5**. However, mirrors in real life have a reflectivity constant R which is the fraction of the incident energy that is reflected, meaning $1 - R$ of the energy will be transmitted (i.e. pass through the mirror). If the initial beam energy between the mirrors is I_0, what is the ratio of the total energy escaping through each mirror?

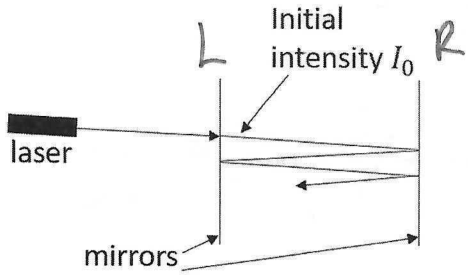


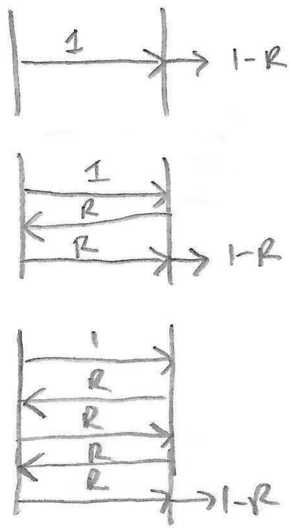
Figure 5: A pair of parallel mirrors with a beam of initial intensity I_0 reflected between them. (The beam has been shown at a small angle for clarity.)

[4]

Fractional energy loss ϕ_R

Right mirror (R)

$$1-R$$
$$+ R^2(1-R)$$
$$+ R^4(1-R)$$
$$+ \dots$$



So $$\phi_R = (1-R)\sum_{n=0}^{\infty}(R^2)^n$$

Geometric series: $a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$

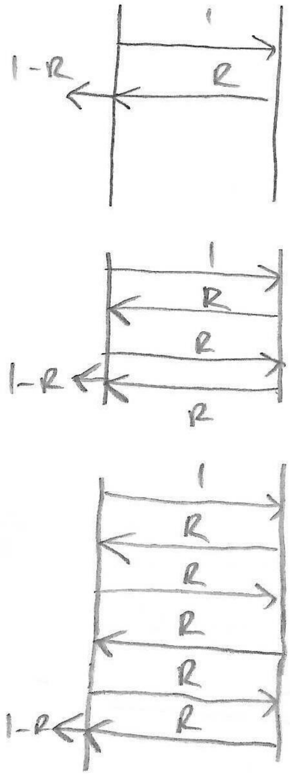
If $|r| < 1$ this is $\frac{a}{1-r}$.

In our case, $a = 1$, $r = R^2$ and $R < 1$

So $$\phi_R = \frac{(1-R)}{1-R^2} = \frac{\cancel{1-R}}{(1+R)\cancel{(1-R)}} = \boxed{\frac{1}{1+R}}$$

Left Mirror (L)

Fractional energy loss ϕ_L



$$\phi_L = (1-R)R\sum_{n=0}^{\infty}(R^2)^n$$

so using $\Phi_R = (1-R)\sum_{n=0}^{\infty}(R^2)^n = \dfrac{1}{1+R}$

$$\Rightarrow \boxed{\phi_L = \frac{R}{1+R}}$$

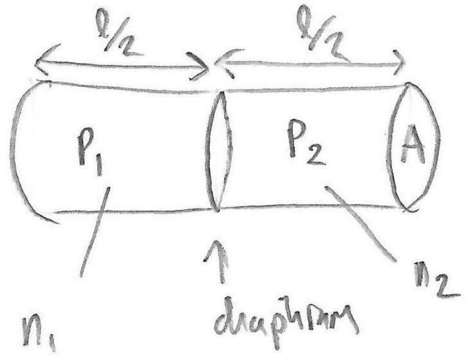
[and $\phi_L / \Phi_R = R$].

i.e. $P_1 > P_2$ initially

r) In a closed tube of length l cross-section area A there is a volume of high pressure gas P_1 and an equal volume of low pressure gas P_2 separated by a diaphragm in the centre of the tube. The tube is uniformly heated from T_0 until the diaphragm bursts, which occurs when a maximum force F is applied. After a moment the pressure equalises in the tube, which is now insulated from the surroundings so that no thermal energy is lost. What is the final pressure inside the tube?

[6]

i.e. temperature remains constant.



Initially:

$$① \quad P_1 A \tfrac{l}{2} = n_1 R T_0$$
$$② \quad P_2 A \tfrac{l}{2} = n_2 R T_0$$

i.e. ideal gases and same initial temperature T_0.

When diaphragm about to burst, when $T = T'$

$$⑤ \quad F = (P_1' - P_2')A$$
$$③ \quad P_1' A \tfrac{l}{2} = n_1 R T'$$
$$④ \quad P_2' A \tfrac{l}{2} = n_2 R T'$$

↳ then diaphram bursts and we have a ideal gas relationship

$$⑥ \quad P_F A l = (n_1 + n_2) R T'$$

Want P_F in terms of A, l, T_0, P_1, P_2

i.e. $P \propto T$ since V, n constant

$$n_1 = \frac{P_1 A l}{2RT_0} \qquad n_2 = \frac{P_2 A l}{2RT_0} \qquad \therefore P_1' = \frac{P_1 \cancel{Al}}{\cancel{2R}T_0} \cancel{R} T' \times \frac{\cancel{2}}{\cancel{Al}} = \frac{P_1 T'}{T_0}$$

$$P_2' = P_2 T' / T_0$$

$$\therefore F/A = \frac{P_1 T'}{T_0} - \frac{P_2 T'}{T_0} \Rightarrow \boxed{T' = \frac{F T_0}{A}(P_1 - P_2)^{-1}}$$

$$\therefore \quad P_F = \frac{n_1+n_2}{A\ell} RT'$$

$$= \frac{A\cancel{\ell}}{2\cancel{R}\cancel{T_0}} (P_1+P_2) \frac{R F \cancel{T_0}}{\cancel{A}} \frac{1}{P_1-P_2} \frac{1}{A\cancel{\ell}}$$

$$\therefore \quad \boxed{P_F = \frac{F}{2A}\left(\frac{P_1+P_2}{P_1-P_2}\right)}$$