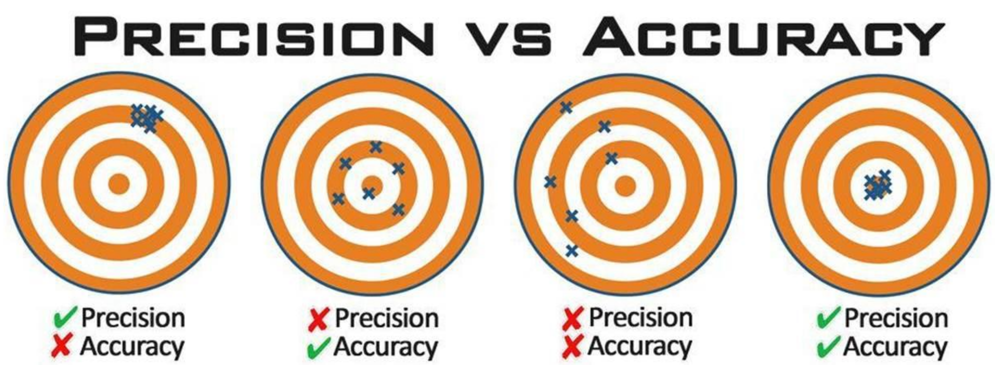
Basic ideas of precision, accuracy and error analysis



$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2}$$

$$x_- \le x \le x_+ \;,\; y_- \le y \le y_+$$

$$x_-^{\,2} y_- \le x^2 y \le x_+^{\,2} y_+ \;\text{ and }\; x_-^{\,2}/y_+ < x^2/y < x_+^{\,2}/y_-$$

$$x = \bar{x} \pm \sigma_x$$

Excel in Physics

Dr Andrew French. 2020. http://www.eclecticon.info/physics_notes.htm

Standard form

Very small and very large quantities are tedious (and error prone) to write out using full decimal notation.

Standard form: e.g. 6.67×10^{-11} is an *integer* between 1 and 9 followed by N - 1 digits, where N is the number of **significant figures** of the quantity.

The power of 10 (the 'exponent') gives you an *immediate sense of scale*.

Precision. A precise measurement is performed to a high number of significant figures. This means the *random error* in the measurement (i.e. the ***standard deviation***) is *very small* compared to the ***mean value***. In calculations, one should quote a answer to the *worst precision* (i.e. lowest number of significant figures) of the *input values.*

$$x = 123.4, \quad y = 56.7, \quad z = 8.9$$

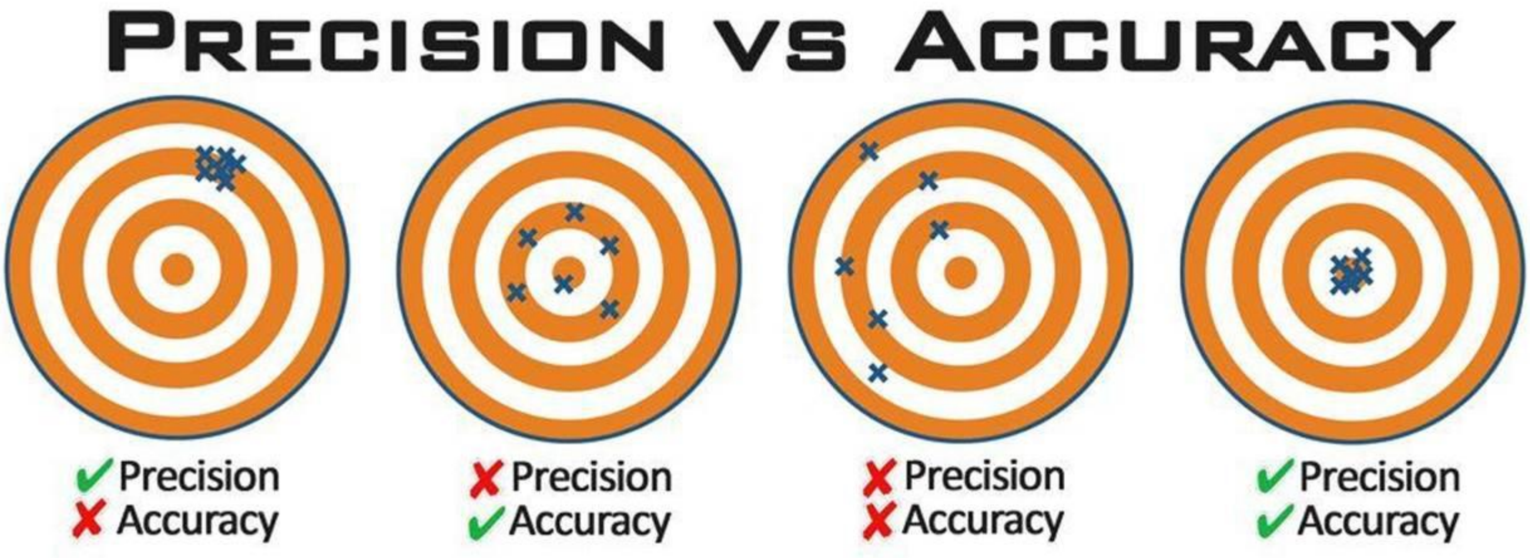
$$\therefore x = 1.234 \times 10^2, \quad y = 5.67 \times 10^1, \quad z = 8.9$$

lowest precision
i.e. 2 s.f.

$$a = \frac{xy}{z} = \frac{123.4 \times 56.7}{8.9} = 786.1550.... \quad \text{(unrounded)}$$

$$a = 7.9 \times 10^2 \quad \text{to 2.s.f}$$

Accuracy relates to the degree of *systematic error*. A time of 12.345s may be *very precise*, but could easily be 2.000s out from a true value of 10.345s if there is some form of accidental offset in the timing system.



Mean and standard deviation

If you have a *sample* of data, which you believe represents a quantity x subject to *random error*:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

is an *unbiased estimator* of the **mean value** of the quantity x.
N is the number of measurements, and x_i is the i^{th} measurement.

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2}$$

is an *unbiased estimator* of the **error** in this measurement. This is *not quite* the *standard deviation*, which involves an N factor rather than N - 1 in the fraction preceding the sum.

The measurement x can therefore be quoted: $x = \bar{x} \pm \sigma_x$

ERROR CALCULATION

ACTUAL X VALUE 123

X VALUES WITH RANDOM ERROR

121	121	125	122	120	128	120	121	124	119

N 10

MEAN X $\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$

122

$$\left(x_i - \overline{x}\right)^2$$

1.21	1.21	8.41	0.01	4.41	34.8	4.41	1.21	3.61	9.61

ERROR IN X $\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \overline{x}\right)^2}$

3

SO: X = (122 +/- 3)

Errors. All measurable quantities will be subject to *uncertainty*. If quantities x,y.... are within a known range, we can use **upper and lower bounds** to determine the range of combined quantities.

e.g. $$x_- \le x \le x_+ \qquad y_- \le y \le y_+$$

Therefore: $$x_-^2 y_- \le x^2 y \le x_+^2 y_+ \qquad x_-^2 / y_+ < x^2 / y < x_+^2 / y_-$$

Note the mixing of *upper and lower bounds* in the last example.

Example: $$1.23 \le x \le 4.56, \;\; 7.89 \le y \le 11.2$$

$$z = \frac{\sqrt{y}}{x}$$

$$\frac{\sqrt{7.89}}{4.56} < z < \frac{\sqrt{11.2}}{1.23}$$

$$\boxed{0.616 < z < 2.721}$$

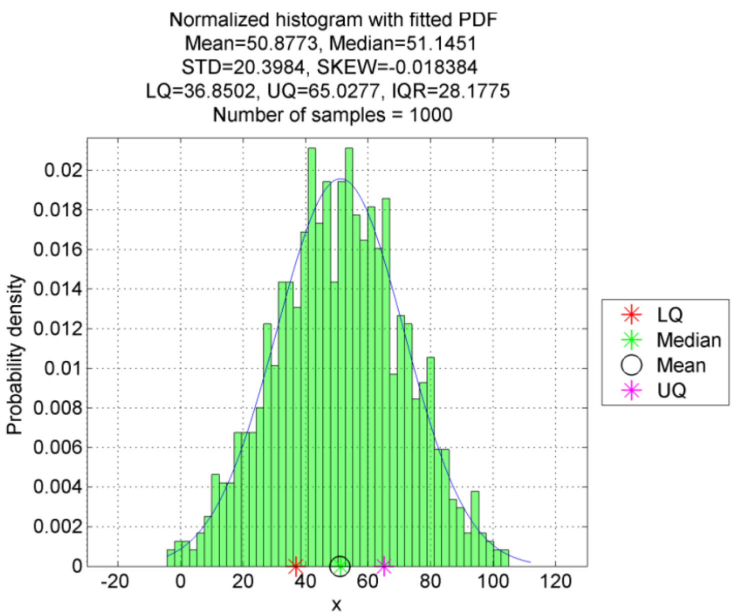
Laws of Errors – but only if you think errors are *normally distributed*

If errors are *normally distributed*, the **'Law of Errors'** can be useful (although may result in an artificially tighter uncertainty than upper and lower bounds). Let $f(x,y,z..)$ be a function of measureable quantities e.g. $x=\bar{x}\pm\sigma_x$.

$$f=\bar{f}\pm\sigma_f \text{ where } \bar{f}=f(\bar{x},\bar{y},\bar{z}...) \quad : \qquad \sigma_f^2=\left(\frac{\partial f}{\partial x}\sigma_x\right)^2+\left(\frac{\partial f}{\partial y}\sigma_y\right)^2+\left(\frac{\partial f}{\partial z}\sigma_z\right)^2+...$$

If $f(x,y....)=kx^a y^b... \Rightarrow \left(\frac{\sigma_f}{f}\right)^2=\left(\frac{a\sigma_x}{x}\right)^2+\left(\frac{b\sigma_y}{y}\right)^2+...$ You *add* the (power weighted) squares of fractional errors.

If a quantity x is subject to random error and N independent measurements $\{x_i\}$ are made, the *unbiased estimate* of the mean value of x is: $\bar{x}=\frac{1}{N}\sum_{i=1}^{N}x_i$. Since the mean value is used in the calculation of the *standard deviation*, the unbiased estimate of the standard deviation in x (i.e. the 'error' in x) is: $\sigma_x=\sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{x})^2}$. We can quote: $x=\bar{x}\pm\sigma_x$



Example:

$$z=3x^2y^{-\frac{1}{2}}, \quad x=20\pm 3, \quad y=40\pm 5$$

$$\therefore \left(\frac{\sigma_z}{\bar{z}}\right)^2=\left(\frac{2\sigma_x}{\bar{x}}\right)^2+\left(\frac{\frac{1}{2}\sigma_y}{\bar{y}}\right)^2$$

$$\bar{z}=3\times 20^2\times 40^{-\frac{1}{2}}=189.7366....$$

$$\therefore \sigma_z=\bar{z}\sqrt{\left(\frac{2\times 3}{20}\right)^2+\left(\frac{\frac{1}{2}5}{40}\right)^2}=58.14....$$

$$\therefore z=(1.9\pm 0.6)\times 10^2$$