$$k=\frac{2\pi}{\lambda}\qquad \omega=2\pi f\qquad \omega=\frac{c}{n}k\qquad \text{wave speed}=\frac{c}{n}\qquad \text{wave vector }\ \mathbf{k}=k\hat{\mathbf{z}}$$

Electromagnetic waves and polarization

Electromagnetic waves (of a particular amplitude, and wavelength) comprise of sinusoidally varying vector components of electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields.
Maxwell's Equations, which describe the relationships between electric and magnetic fields (and charge) predict the following:

1. If an electromagnetic wave propagates in direction parallel to vector $\mathbf{k}$, the electric and magnetic field are both *perpendicular* to this direction. In other words ($\mathbf{E},\mathbf{B},\boldsymbol{k}$) forms a right handed set* in a Cartesian (x,y,z) sense. No vector component of $\mathbf{E}$ or $\mathbf{B}$ is parallel to the direction of propagation.

2. Electromagnetic waves travel at a *finite speed* through a medium. This is independent of any coordinate system, so you can never 'catch up' with an electromagnetic wave, no matter how fast you move. This idea is the main reason (in *Special Relativity*) behind the need to modify space and time as one approaches the speed of light. The speed of electromagnetic waves is c/n where $c = 2.998 \times 10^8\,\text{ms}^{-1}$ and n is the refractive index. For a vacuum, n is unity. A magnitude less than unity is impossible*

3. At an interface between media of differing refractive index, vector components of $\mathbf{B}$ *perpendicular* to the interface surface must be *continuous* across the boundary. Also, components of the $\mathbf{E}$ and $\mathbf{H}$ fields which are *parallel* to the surface, must be continuous across the boundary.

4. For an electromagnetic wave: $|\mathbf{E}|=|\mathbf{B}|\dfrac{c}{n}$ $\qquad \mathbf{B}=\hat{\mathbf{k}}\times\dfrac{n\mathbf{E}}{c}$ $\qquad n=\sqrt{\mu\varepsilon}$

μ **Relative permeability.** Unity for non magnetic materials. Magnetic materials such as iron have a relative permeability of about 5000. Ferrite is 640, Nickel is 100.
ε **Relative permittivity.** Unity for vacuum and approximately for air. Water is about 1.77, glass 3.7-10, diamond 5.5-10, sapphire 8.9-11.1

$$c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}=2.998\times10^{8}\,\text{ms}^{-1}$$

$$\mu_0=4\pi\times10^{-7}\,\text{Hm}^{-1}\qquad \varepsilon_0=8.85\times10^{-12}\,\text{Fm}^{-1}$$

For isotropic media:

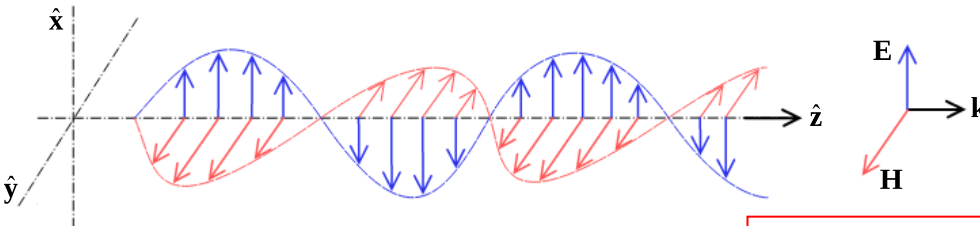
$$\mathbf{B}=\mu_0\mu\mathbf{H}\qquad \mathbf{H}=\frac{n}{c\mu\mu_0}\begin{pmatrix}-E_{0y}\\E_{0x}\\0\end{pmatrix}e^{i(kz-\omega t)}$$

$$\mathbf{S}=\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

Poynting vector
i.e. Power / m²

$$\therefore |\mathbf{S}|_{\max}=\frac{1}{\mu_0}|\mathbf{E}||\mathbf{B}|=\frac{n}{\mu_0 c}|\mathbf{E}|^2$$

$$\langle S\rangle\approx\tfrac{1}{2}\frac{n}{377\Omega}|\mathbf{E}|^2$$



$$\mathbf{E}=\begin{pmatrix}E_{0x}\\E_{0y}\\0\end{pmatrix}e^{i(kz-\omega t)}\qquad \mathbf{B}=\frac{n}{c}\begin{pmatrix}-E_{0y}\\E_{0x}\\0\end{pmatrix}e^{i(kz-\omega t)}$$

Power of an EM wave varies as the *square* of electric field strength

Energy per unit volume stored in $\mathbf{E}$ and $\mathbf{B}$ fields is: $u=\tfrac{1}{2}\varepsilon_0|\mathbf{E}|^2+\tfrac{1}{2}\dfrac{1}{\mu_0}|\mathbf{B}|^2$

Note the actual $\mathbf{E}$ and $\mathbf{B}$ fields are the *real* parts of these complex quantities.

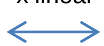
The ***polarization*** of an electromagnetic wave describes the relationship between the electric field vector components and how they vary with time t and propagation distance z.

$$\mathbf{E}=E_0e^{i\phi_0}\begin{pmatrix}a\\be^{i\delta}\end{pmatrix}e^{i(kz-\omega t)}\qquad \text{i.e.}\quad \frac{E_{0y}}{E_{0x}}=\frac{be^{i\delta}}{a}$$

$\begin{pmatrix}a\\be^{i\delta}\end{pmatrix}$ is called the **Jones vector.** Different values of a,b and phase δ give rise to *linear*, *circular* and *elliptical* polarizations.
This is because the *time variation* of the electric field vector.

$\text{Re}(\mathbf{E})=\text{Re}\left(\begin{pmatrix}a\\be^{i\delta}\end{pmatrix}e^{-i\omega t}\right)$ will follow a linear, circular or elliptical trajectory depending on the values set.

$\begin{pmatrix}1\\0\end{pmatrix}$ x linear



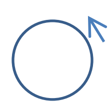
$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ Linear 45°

Note *De Moivre's Theorem*

$$e^{i\theta}=\cos\theta+i\sin\theta$$
$$\Rightarrow e^{i\frac{1}{2}\pi}=i\qquad e^{i\pi}+1=0$$

$\begin{pmatrix}0\\1\end{pmatrix}$ y linear

$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ Left hand circular

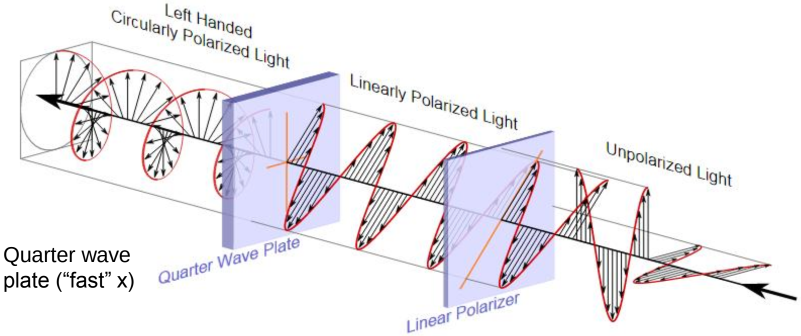


$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$ Right hand circular

**But it is possible to have a *negative* or indeed *complex* refractive index! (*metamaterials* & metals respectively).

*Actually, it is ($\mathbf{E},\mathbf{H},\mathbf{k}$). But for isotropic media, $\mathbf{B}$ is parallel to $\mathbf{H}$

The effect of an electromagnetic wave passing through a **polariser** (e.g. a material which will modify the wave in different ways depending on the polarisation of the electric field) can be modelled by the *matrix multiplication* of the Jones vector **J** for an incident wave by a 2 x 2 **Jones matrix**.



$$\mathbf{J} = \begin{pmatrix} a \\ be^{i\delta} \end{pmatrix}$$

$$\mathbf{J} \to \mathbf{MJ}$$

$$\mathbf{E} = E_0 e^{i\phi_0} \begin{pmatrix} a \\ be^{i\delta} \end{pmatrix} e^{i(kz-\omega t)} = E_0 e^{i\phi_0} \mathbf{J} e^{i(kz-\omega t)}$$

$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ Linear // x

$\mathbf{M} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ Linear //y

$\mathbf{M} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ Linear 45º

$\mathbf{M} = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ Linear -45º

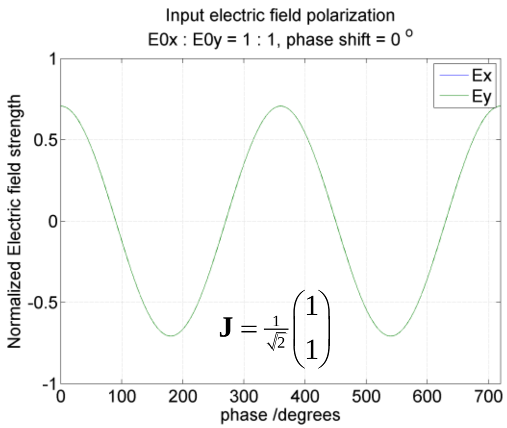
$\mathbf{M} = \tfrac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ Right circular

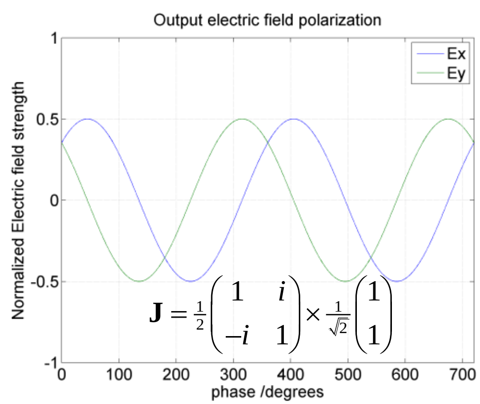
$\mathbf{M} = \tfrac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ Left circular

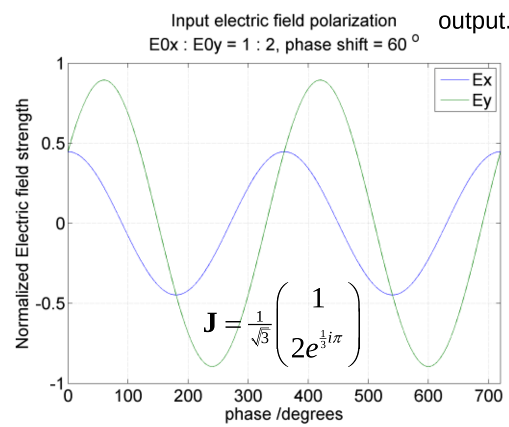
$\mathbf{M} = e^{\frac{1}{4}i\pi}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ Quarter wave plate (“fast” x)

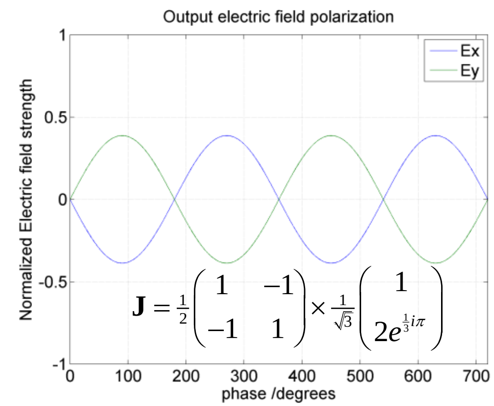
$\mathbf{M} = e^{\frac{1}{4}i\pi}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ Quarter wave plate (“fast” y)

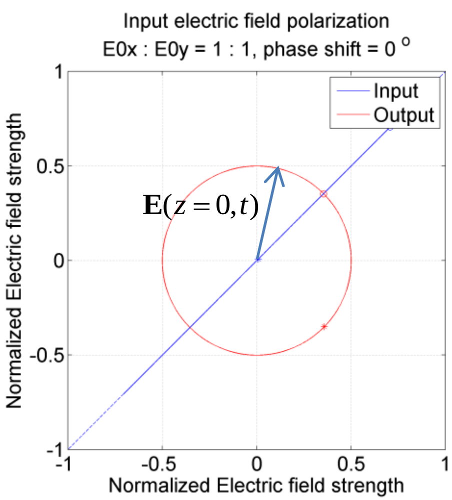
The **Jones matrix** defines the output polarization. The amount of transmission loss depends on how close the original polarization was to the desired output.











$$\mathbf{J} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

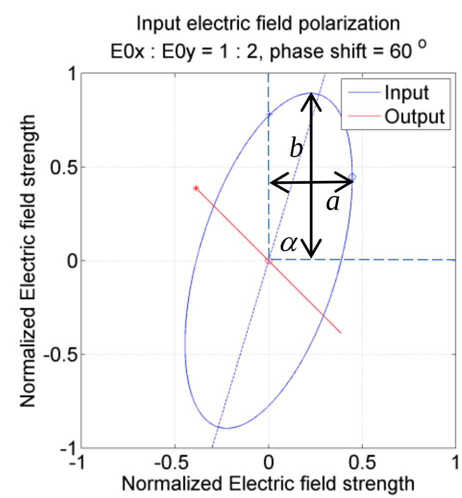
$$\mathbf{M} = \tfrac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

In this example, 45º linear polarisation becomes Right circular following application of the polariser

For **elliptical polarization**, the tilt of the ellipse (α) is given by

$$\tan 2\alpha = \frac{2ab}{a^2 - b^2}\cos\delta$$

In this example, elliptical polarisation becomes 45º linear following application of the polariser.

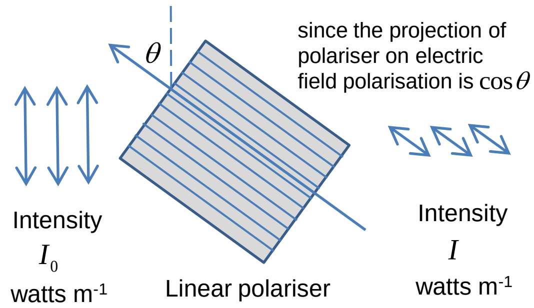


$$\mathbf{J} = \tfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 2e^{\frac{1}{3}i\pi} \end{pmatrix}$$

$$\mathbf{M} = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Note if linear polarised light is incident upon a linear polariser with polarisation direction tilted by θ from the polarisation of the incident light

Malus's Law states $I = I_0 \cos^2\theta$



The Fresnel Equations

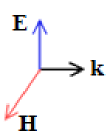
If an electromagnetic wave meets a change in refractive index, the general response will be for **reflected** and **transmitted** (i.e. ***refracted***) waves to be created. The *power* of the **incident** wave will be shared between these. The balance of power depends on a number of factors, which are modelled via the *Fresnel Equations.*

$$\mathbf{E} = \begin{pmatrix} E_{0x} \\ E_{0y} \\ 0 \end{pmatrix} e^{i(kz-\omega t)}$$

$$\mathbf{B} = \frac{n}{c}\begin{pmatrix} -E_{0y} \\ E_{0x} \\ 0 \end{pmatrix} e^{i(kz-\omega t)}$$

$$\mathbf{H} = \frac{n}{c\mu\mu_0}\begin{pmatrix} -E_{0y} \\ E_{0x} \\ 0 \end{pmatrix} e^{i(kz-\omega t)}$$

Let us start with the $\mathbf{E},\mathbf{B}$ and $\mathbf{H}$ fields associated with an electromagnetic wave in *isotropic* media (see previous pages)



Augustin-Jean Fresnel
1788-1927

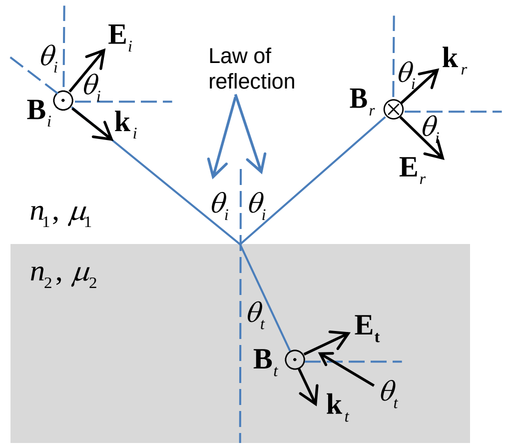
Maxwell's Equations tell us: "At an interface between media of differing refractive index, vector components of $\mathbf{B}$ *perpendicular* to the interface surface must be *continuous* across the boundary. Also, components of the $\mathbf{E}$ and $\mathbf{H}$ fields which are *parallel* to the surface, must be continuous across the boundary."

Let us consider two scenarios separately:

Case 1: Electric field vector is **parallel** to the plane containing the incident, reflected and transmitted wave propagation directions

$$|\mathbf{E}_i| = E_i \quad |\mathbf{E}_r| = E_r \quad |\mathbf{E}_t| = E_t$$

$$\mathbf{H} = \frac{1}{\mu\mu_0}\mathbf{B} \quad |\mathbf{B}_i| = \frac{n_1 E_i}{c} \quad |\mathbf{B}_r| = \frac{n_1 E_r}{c} \quad |\mathbf{B}_t| = \frac{n_2 E_t}{c}$$



$\mathbf{E}_{\parallel}$ continuity: $\quad E_i \cos\theta_i + E_r \cos\theta_i = E_t \cos\theta_t$

$\mathbf{H}_{\parallel}$ continuity: $\quad \frac{n_1}{\mu_1\mu_0 c}E_i - \frac{n_1}{\mu_1\mu_0 c}E_r = \frac{n_2}{\mu_2\mu_0 c}E_t$

$$\therefore E_t = \frac{n_1\mu_2}{n_2\mu_1}(E_i - E_r)$$

$$\therefore E_i \cos\theta_i + E_r \cos\theta_i = \frac{n_1\mu_2}{n_2\mu_1}(E_i - E_r)\cos\theta_t$$

$$E_i\left(-\cos\theta_i + \frac{n_1\mu_2}{n_2\mu_1}\cos\theta_t\right) = E_r\left(\cos\theta_i + \frac{n_1\mu_2}{n_2\mu_1}\cos\theta_t\right)$$

$$E_i\left(\frac{n_1}{\mu_1}\cos\theta_t - \frac{n_2}{\mu_2}\cos\theta_i\right) = E_r\left(\frac{n_2}{\mu_2}\cos\theta_i + \frac{n_1}{\mu_1}\cos\theta_t\right)$$

$$r_{\parallel} = \frac{E_r}{E_i} = \frac{\frac{n_1}{\mu_1}\cos\theta_t - \frac{n_2}{\mu_2}\cos\theta_i}{\frac{n_2}{\mu_2}\cos\theta_i + \frac{n_1}{\mu_1}\cos\theta_t}$$

$$E_t = \frac{n_1\mu_2}{n_2\mu_1}(E_i - E_r)$$

$$t_{\parallel} = \frac{E_t}{E_i} = \frac{n_1\mu_2}{n_2\mu_1}(1 - r_{\parallel})$$

$$t_{\parallel} = \frac{n_1\mu_2}{n_2\mu_1}\left(1 - \frac{\frac{n_1}{\mu_1}\cos\theta_t - \frac{n_2}{\mu_2}\cos\theta_t}{\frac{n_2}{\mu_2}\cos\theta_i + \frac{n_1}{\mu_1}\cos\theta_t}\right)$$

$$t_{\parallel} = \frac{n_1\mu_2}{n_2\mu_1}\left(\frac{\frac{2n_2}{\mu_2}\cos\theta_i}{\frac{n_2}{\mu_2}\cos\theta_i + \frac{n_1}{\mu_1}\cos\theta_t}\right)$$

$$t_{\parallel} = \frac{\frac{2n_1}{\mu_1}\cos\theta_i}{\frac{n_2}{\mu_2}\cos\theta_i + \frac{n_1}{\mu_1}\cos\theta_t}$$

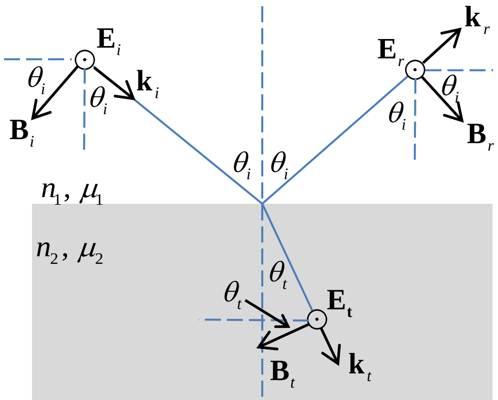
Power coefficients are:

$$R_{\parallel} = |r_{\parallel}|^2$$

$$T_{\parallel} = 1 - |r_{\parallel}|^2$$

This scenario is commonly known as 'P' polarisation (P for Parallel)

Case 2: Electric field vector is **perpendicular** to the plane containing the incident, reflected and transmitted wave propagation directions



$$E_i + E_r = E_t$$

$$t_\perp = \frac{E_t}{E_i} = 1 + r_\perp$$

$$t_\perp = 1 + \frac{\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}$$

$$t_\perp = \frac{\frac{2n_1}{\mu_1}\cos\theta_i}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}$$

This scenario is commonly known as '**S**' polarisation (S from *senkrecht*, which is German for perpendicular)

Power coefficients are:

$$R_\perp = |r_\perp|^2$$

$$T_\perp = 1 - |r_\perp|^2$$

$$\mathbf{H} = \frac{1}{\mu\mu_0}\mathbf{B} \quad |\mathbf{E}_i| = E_i \quad |\mathbf{E}_r| = E_r \quad |\mathbf{E}_t| = E_t$$

$$|\mathbf{B}_i| = \frac{n_1 E_i}{c} \quad |\mathbf{B}_r| = \frac{n_1 E_r}{c} \quad |\mathbf{B}_t| = \frac{n_2 E_t}{c}$$

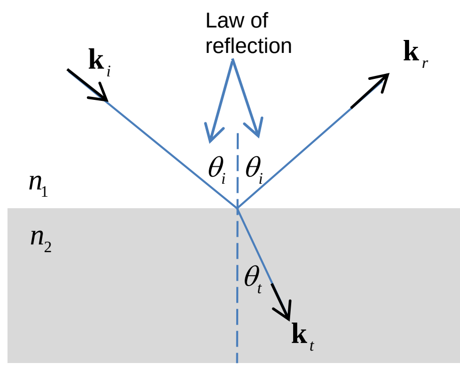
$\mathbf{E}_{\parallel}$ continuity: $E_i + E_r = E_t$

$\mathbf{H}_{\parallel}$ continuity: $\frac{n_1}{\mu_1\mu_0 c}E_i\cos\theta_i - \frac{n_1}{\mu_1\mu_0 c}E_r\cos\theta_i = \frac{n_2}{\mu_2\mu_0 c}E_t\cos\theta_t$

$$\therefore \frac{n_1}{\mu_1}E_i\cos\theta_i - \frac{n_1}{\mu_1}E_r\cos\theta_i = \frac{n_2}{\mu_2}(E_i + E_r)\cos\theta_t$$

$$E_i\left(\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t\right) = E_r\left(\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t\right)$$

$$r_\perp = \frac{\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}$$



$$0 < \frac{n_1\sin\theta_i}{n_2} < 1$$ *Always true* if $n_2 > n_1$

$n_1 > n_2$

$$\theta_i < \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Critical angle

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Snell's Law of refraction

$$n_1\sin\theta_i = n_2\sin\theta_t$$

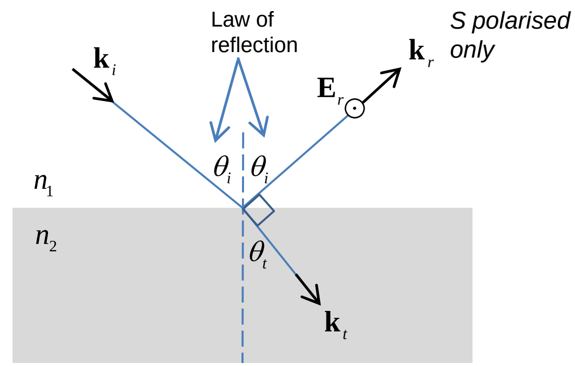
$$\therefore \theta_t = \sin^{-1}\left(\frac{n_1\sin\theta_i}{n_2}\right)$$

Real solutions for the transmitted ray angle when

$$0 < \frac{n_1\sin\theta_i}{n_2} < 1$$

So EM waves propagating from a high refractive index medium to a lower refractive index medium will be **internally reflected** (i.e. no transmission) when the angle of incidence exceeds a ***critical angle*** θ_c

Brewster's Angle



If a scenario arises where the angle between reflected and transmitted waves is a right angle, this means *the reflected waves can only be S polarised.*

P-polarised transmitted waves result from electrons oscillating parallel to the plane at the interface of the two mediums. However, *no radiation occurs in the direction of he polarisation.* **Since the *transmitted* wave polarisation direction is parallel to the *reflected* wavevector**, this means *no P polarised radiation is reflected.*

$\theta_i + 90^\circ + \theta_t = 180^\circ$ From the geometry of the above diagram

$\therefore \theta_t = 90^\circ - \theta_i$

$n_1\sin\theta_i = n_2\sin\theta_t$ Snell's Law

$\therefore n_1\sin\theta_i = n_2\sin(90^\circ - \theta_i)$

$n_1\sin\theta_i = n_2\cos\theta_i$

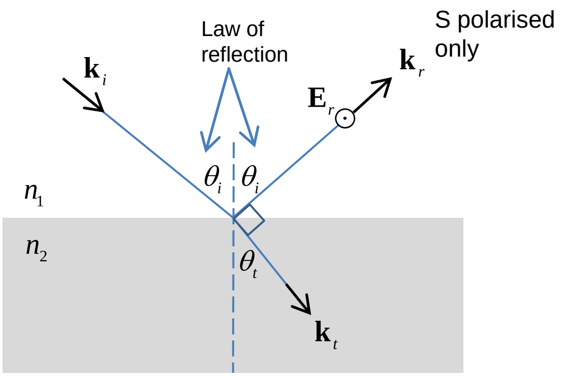
$\tan\theta_i = \frac{n_2}{n_1}$

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right)$$ Brewster's Angle

Sir David Brewster 1781-1869

This effect is used in the design of glare reducing optical devices such as sunglasses, car windows and photographic lens filters.

Brewster's Angle from the Fresnel equations



For P-polarised EM waves

$$r_{\parallel} = \frac{E_r}{E_i} = \frac{\frac{n_1}{\mu_1}\cos\theta_t - \frac{n_2}{\mu_2}\cos\theta_i}{\frac{n_2}{\mu_2}\cos\theta_i + \frac{n_1}{\mu_1}\cos\theta_t}$$

Assuming the non-magnetic media

$$\mu_1 = \mu_2 = 1$$

$$\therefore r_{\parallel} = 0 \Rightarrow n_1 \cos\theta_t - n_2 \cos\theta_i = 0$$

$$n_1 \cos\theta_t = n_2 \cos\theta_i$$

$$\left(\frac{n_1}{n_2}\right)^2 \cos^2\theta_t = \cos^2\theta_i$$

$$\left(\frac{n_1}{n_2}\right)^2 \left(1 - \sin^2\theta_t\right) = \cos^2\theta_i$$

Snell's Law

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$\therefore \sin^2\theta_t = \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i$$

$$\left(\frac{n_1}{n_2}\right)^2 \left(1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i\right) = \cos^2\theta_i$$

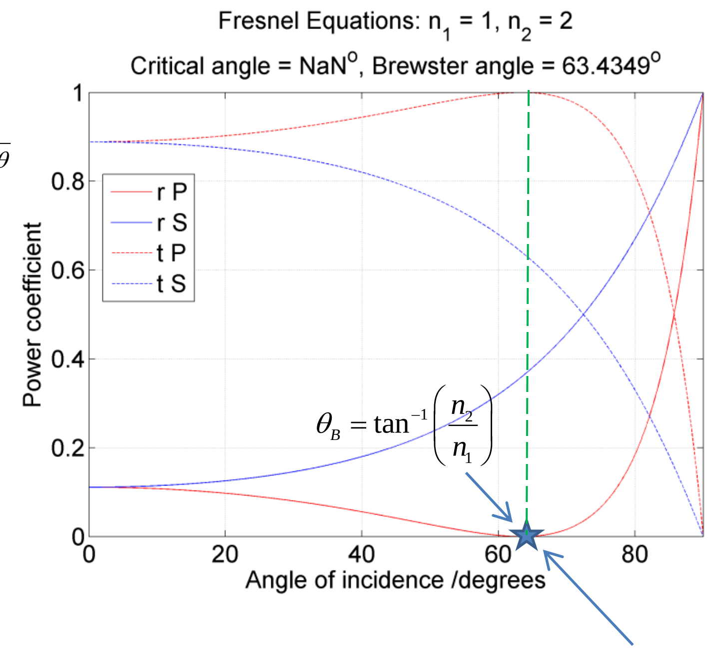
$$\left(\frac{n_1}{n_2}\right)^2 \left(\frac{1}{\cos^2\theta_i} - \left(\frac{n_1}{n_2}\right)^2 \tan^2\theta_i\right) = 1 \qquad 1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

$$\left(\frac{n_1}{n_2}\right)^2 \left(1 + \tan^2\theta_i - \left(\frac{n_1}{n_2}\right)^2 \tan^2\theta_i\right) = 1$$

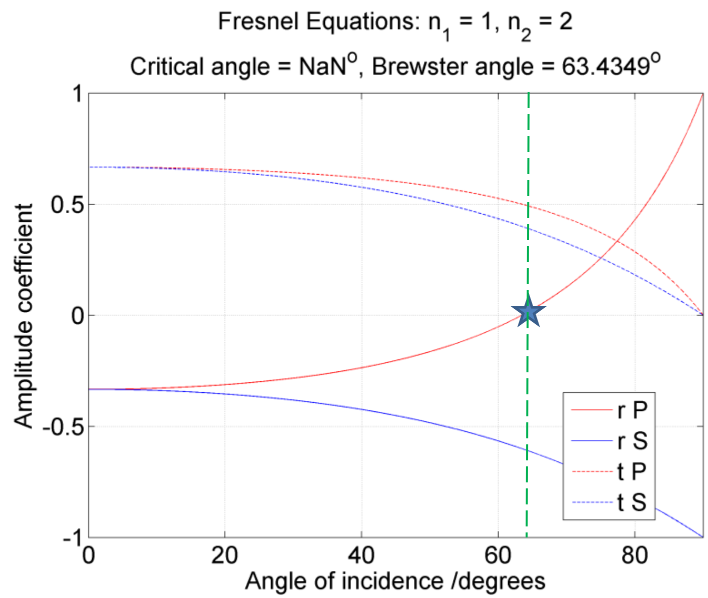
$$\left(1 - \left(\frac{n_1}{n_2}\right)^2\right) \tan^2\theta_i = \left(\frac{n_2}{n_1}\right)^2 - 1$$

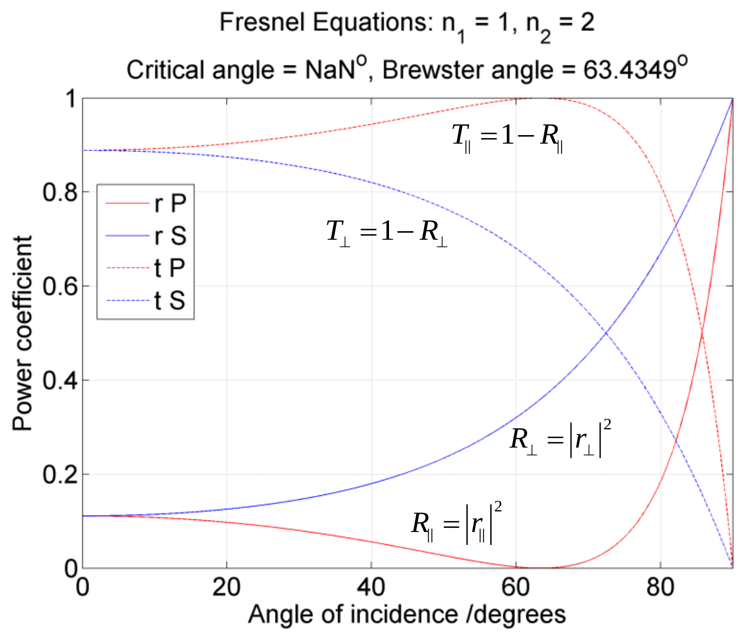
$$\left(\frac{n_2^2 - n_1^2}{n_2^2}\right) \tan^2\theta_i = \left(\frac{n_2^2 - n_1^2}{n_1^2}\right)$$

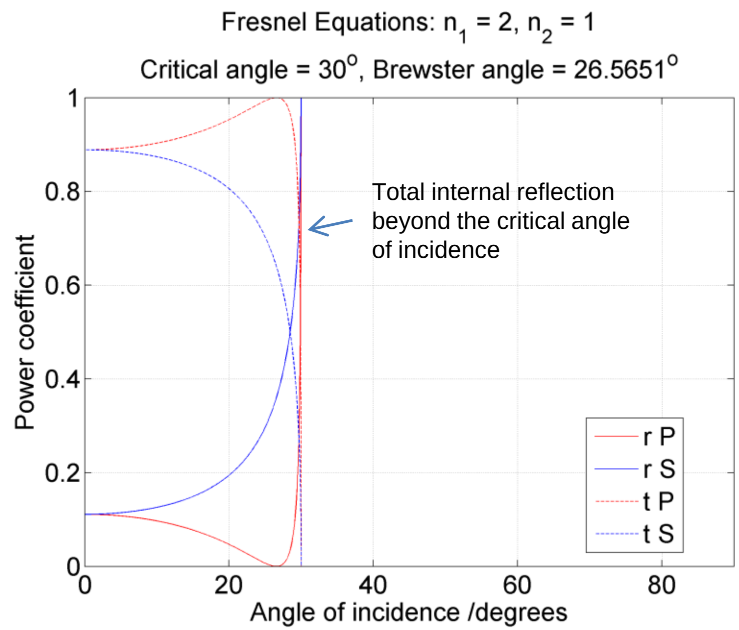
$$\tan\theta_i = \frac{n_2}{n_1}$$



The power coefficient for P-polarised is zero (and also a minima) at the Brewster angle.

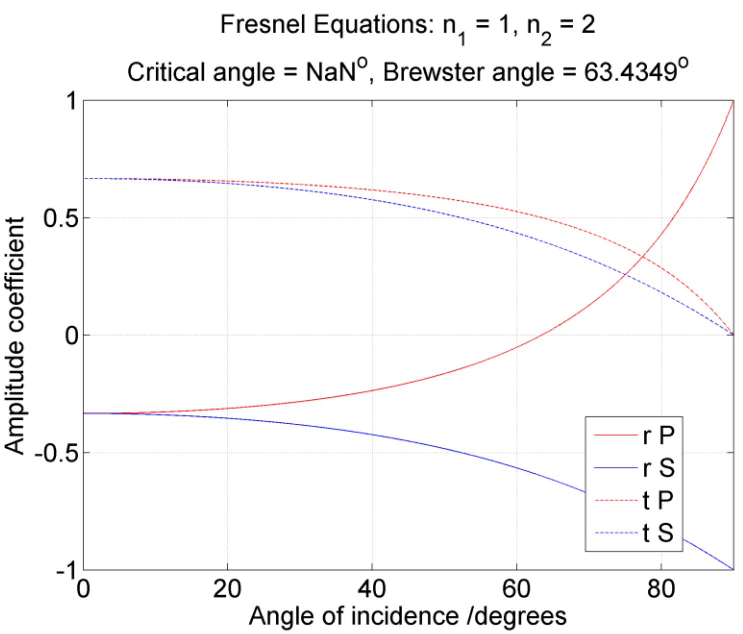


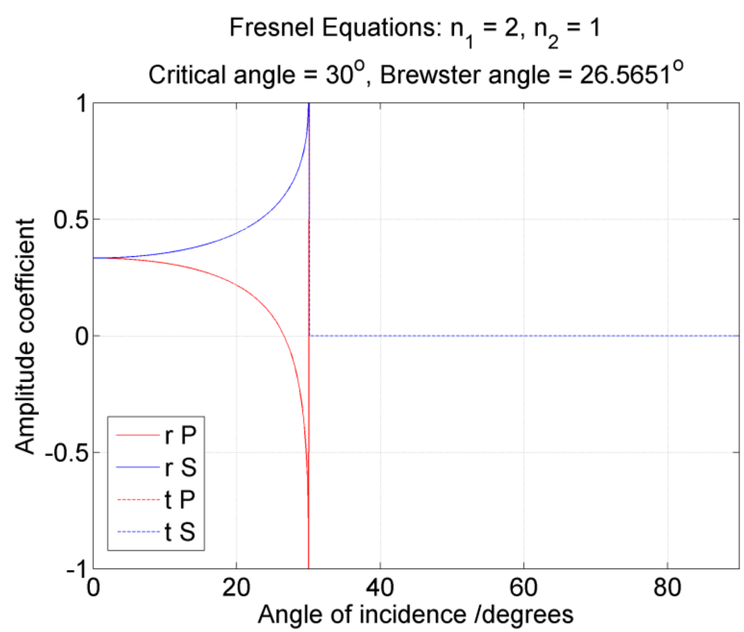




$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right)$$





P-polarised Electric field (*parallel* to plane containing wave vectors)

$$r_\parallel = \frac{E_r}{E_i} = \frac{\frac{n_1}{\mu_1}\cos\theta_t - \frac{n_2}{\mu_2}\cos\theta_i}{\frac{n_2}{\mu_2}\cos\theta_i + \frac{n_1}{\mu_1}\cos\theta_t}$$

$$t_\parallel = \frac{\frac{2n_1}{\mu_1}\cos\theta_i}{\frac{n_2}{\mu_2}\cos\theta_i + \frac{n_1}{\mu_1}\cos\theta_t}$$

S-polarised Electric field (*perpendicular* to plane containing wave vectors)

$$r_\perp = \frac{\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}$$

$$t_\perp = \frac{\frac{2n_1}{\mu_1}\cos\theta_i}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}$$