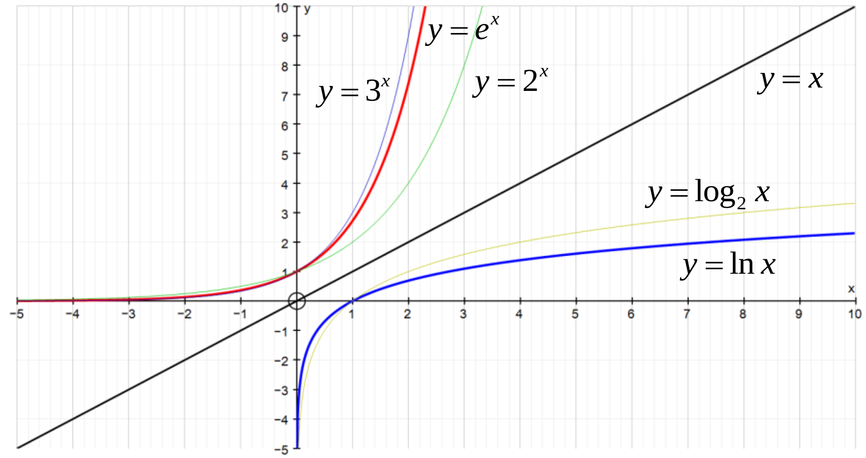
Logarithms (and exponentials)

Logarithms are the *inverse functions* of *exponentials*, which take the form: $y = b^x$

Note: $b > 0$ e.g $y = 2^x$ $y = \left(\frac{1}{3}\right)^x$ $y = \left(\sqrt{5}\right)^x$

b is called the *base* and x is the *exponent*. The logarithm to base *b* is the *power* we raise *b* by to get *y*. i.e.

$\log_b y = x$ e.g $10^3 = 1000$ $\therefore \log_{10} 1000 = 3$ $2^{10} = 1024$ $\therefore \log_2 1024 = 10$ $\left(\sqrt{2}\right)^4 = 4$ $\therefore \log_{\sqrt{2}} 4 = 4$



The algebraic properties of logarithms can be deduced from their exponential forms

Addition

$x = b^A \Rightarrow A = \log_b x$

$y = b^B \Rightarrow B = \log_b y$

$xy = b^{A+B} \Rightarrow A + B = \log_b xy$

$\therefore \log_b xy = \log_b x + \log_b y$

Subtraction

$x = b^A \Rightarrow A = \log_b x$

$y = b^B \Rightarrow B = \log_b y$

$\frac{x}{y} = b^{A-B} \Rightarrow A - B = \log_b \left(\frac{x}{y}\right)$

$\therefore \log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

Powers

$x = b^A \Rightarrow A = \log_b x$

$x^n = b^{nA} \Rightarrow nA = \log_b x^n$

$\therefore \log_b x^n = n \log_b x$

Converting bases

$x = b^A \Rightarrow A = \log_b x$

$\log_c x = A \log_c b \Rightarrow A = \frac{\log_c x}{\log_c b}$

$\therefore \log_b x = \frac{\log_c x}{\log_c b}$

e.g. $\log_{10} 1000 = \frac{\ln 1000}{\ln 10}$

Log as a power

$y = b^{\log_b x}$

$\therefore \log_b y = \log_b x$

$\therefore b^{\log_b x} = x$

Use this to make *any* quantity an exponential using *any* (positive) base

Example application

What are the first twelve digits of 2^{100}?

$2^{100} = 10^{\log_{10} 2^{100}}$

$2^{100} = 10^{100 \log_{10} 2}$

$2^{100} = 10^{30.1029995664}$

$2^{100} = 10^{0.1029995664} \times 10^{30}$

$2^{100} = 1.26765060023 \times 10^{30}$

Note: $\ln x = \log_e x$, $e = 2.71828182846...$ *Natural* logarithm

Misc examples

$\log_x 3 = 2, \quad x > 0$

$x^2 = 3 \quad \therefore x = \sqrt{3}$

$\log_{42} 10 = \frac{1}{\log_{10} 42} = \frac{\ln 10}{\ln 42} = \frac{\log_b 10}{\log_b 42}$

$42 = e^{3x}$

$\ln 42 = 3x$

$\frac{\ln 42}{3} = x$

$3^{-x+1} = 7^{4x+2}$

$(-x+1)\log 3 = (4x+2)\log 7$

$\log 3 - 2\log 7 = x(4\log 7 + \log 3)$

$\frac{\log 3 - 2\log 7}{4\log 7 + \log 3} = x$

$\frac{\log 3 - \log 49}{\log 2401 + \log 3} = x$

$\frac{\log \frac{3}{49}}{\log 7203} = x$