Recipe for inverting 2 x 2 matrices

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

Identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Inverse of a 2x2 matrix

$$\mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$ad - bc$ is called the **determinant** of the 2 x 2 matrix.
If it equals zero there can be no inverse.
A matrix with a determinant of zero is called a *singular* matrix.

e.g.

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}^{-1} = \frac{1}{(1)(4) - (2)(-3)}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} = \tfrac{1}{10}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{pmatrix}$$

Recipe for inverting 3 x 3 matrices

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^T$$

1. Calculate the determinant using the *minors of elements* in the top row. If $\det \mathbf{A} = 0$ *then there is no inverse.*

$$\det \mathbf{A} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Determinant of a 3x3 matrix

Note the minus sign!

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

The **minor** of an element is found by *deleting the row and column in which the element lies, and computing the determinant of the remaining matrix.*

In this case the minor associated with element a is

$$\begin{vmatrix} e & f \\ h & i \end{vmatrix}$$

2. Calculate a matrix of minors

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & \begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & \begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

3. Transform the *matrix of minors* **M** into a **Cofactor matrix C** by multiplying elements by a + - +… alternate sign pattern

$$\mathbf{C} = \begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

4. **Transpose Cofactor matrix** (i.e. swap rows and columns), and divide by $\det \mathbf{A}$ to determine the inverse of $\mathbf{A}$.

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^T = \frac{1}{\det \mathbf{A}}\begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

Example inversion of a 3x3 matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

1. Calculate the determinant using the *minors of elements* in the top row.
   If $\det \mathbf{A}$ = 0 *then there is no inverse.*

$$\det \mathbf{A} = 3\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} - 7\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}$$
$$= 3(6-2) - 2(-2-0) - 7(2-0)$$
$$= 12 + 4 - 14$$
$$= 2$$

2. Calculate a matrix of minors

$$\mathbf{M} = \begin{pmatrix} 4 & -2 & 2 \\ 10 & -6 & 6 \\ -19 & 10 & -11 \end{pmatrix}$$

3. Transform the *matrix of minors* **M** into a **Cofactor matrix C** by multiplying elements by a + - +… alternate sign pattern

$$\mathbf{C} = \begin{pmatrix} 4 & 2 & 2 \\ -10 & -6 & -6 \\ -19 & -10 & -11 \end{pmatrix}$$

4. **Transpose Cofactor matrix** (i.e. swap rows and columns), and divide by $\det \mathbf{A}$ to determine the inverse of $\mathbf{A}$.

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^T = \tfrac{1}{2}\begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix}$$

$$\therefore \mathbf{A}^{-1} = \begin{pmatrix} 2 & -5 & -9.5 \\ 1 & -3 & -5 \\ 1 & -3 & -5.5 \end{pmatrix}$$

Example inversion of a 3x3 matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

1. Calculate the determinant using the *minors of elements* in the top row.
   If $\det \mathbf{A}$ = 0 *then there is no inverse.*

$$\det \mathbf{A} = 2\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}$$
$$= 2(-2-1) - 3(3-2) + 2(3+4)$$
$$= -6 - 3 + 14$$
$$= 5$$

2. Calculate a matrix of minors

$$\mathbf{M} = \begin{pmatrix} 4 & -2 & 2 \\ 10 & -6 & 6 \\ -19 & 10 & -11 \end{pmatrix}$$

3. Transform the *matrix of minors* **M** into a **Cofactor matrix C** by multiplying elements by a + - +… alternate sign pattern

$$\mathbf{C} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

4. **Transpose Cofactor matrix** (i.e. swap rows and columns), and divide by $\det \mathbf{A}$ to determine the inverse of $\mathbf{A}$.

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^T$$

$$= \tfrac{1}{5}\begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

Solving systems of simultaneous equations

$$x + 2y - z = 5$$
$$2x - 3y + z = -1$$
$$-x + y + 2z = 3$$

We can write a set of simultaneous linear equations as a matrix multiplication, which can be solved by multiplying by the inverse.

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ -1 & 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2.0625 \\ 2.1875 \\ 1.4375 \end{pmatrix}$$

This was performed using MATLAB's recipe for computing the inverse of the 3 x 3 matrix

MATLAB ('MATrix LABoratory') is a computer programming language which is based upon matrix algebra.

MATLAB

In general, express a set of three simultaneous linear equations as:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

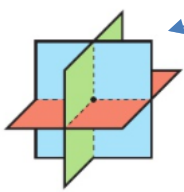
$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

This generalizes To N equations in N unknowns. But we will need a *numerical method* to perform the matrix inversion!

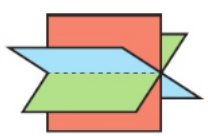
If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \neq 0$

i.e. determinant of 'equation coefficient Matrix' is non zero.

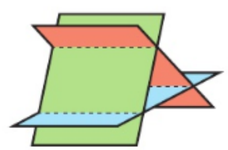
Otherwise … Matrix is **SINGULAR**



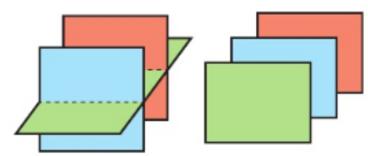
The planes meet at a **point**. The system of equations is **consistent** and has **one solution** represented by this point. This is the only case when the corresponding matrix is **non-singular**.



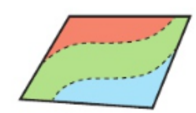
The planes form a **sheaf**. The system of equations is **consistent** and has **infinitely many solutions** represented by the line of intersection of the three planes.



The planes form a **prism**. The system of equations is **inconsistent** and has **no solutions**.



Two or more of the planes are parallel and non-identical. The system of equations is **inconsistent** and has **no solutions**.



All three equations represent the same plane. In this case the system of equations is **consistent** and has **infinitely many solutions**.